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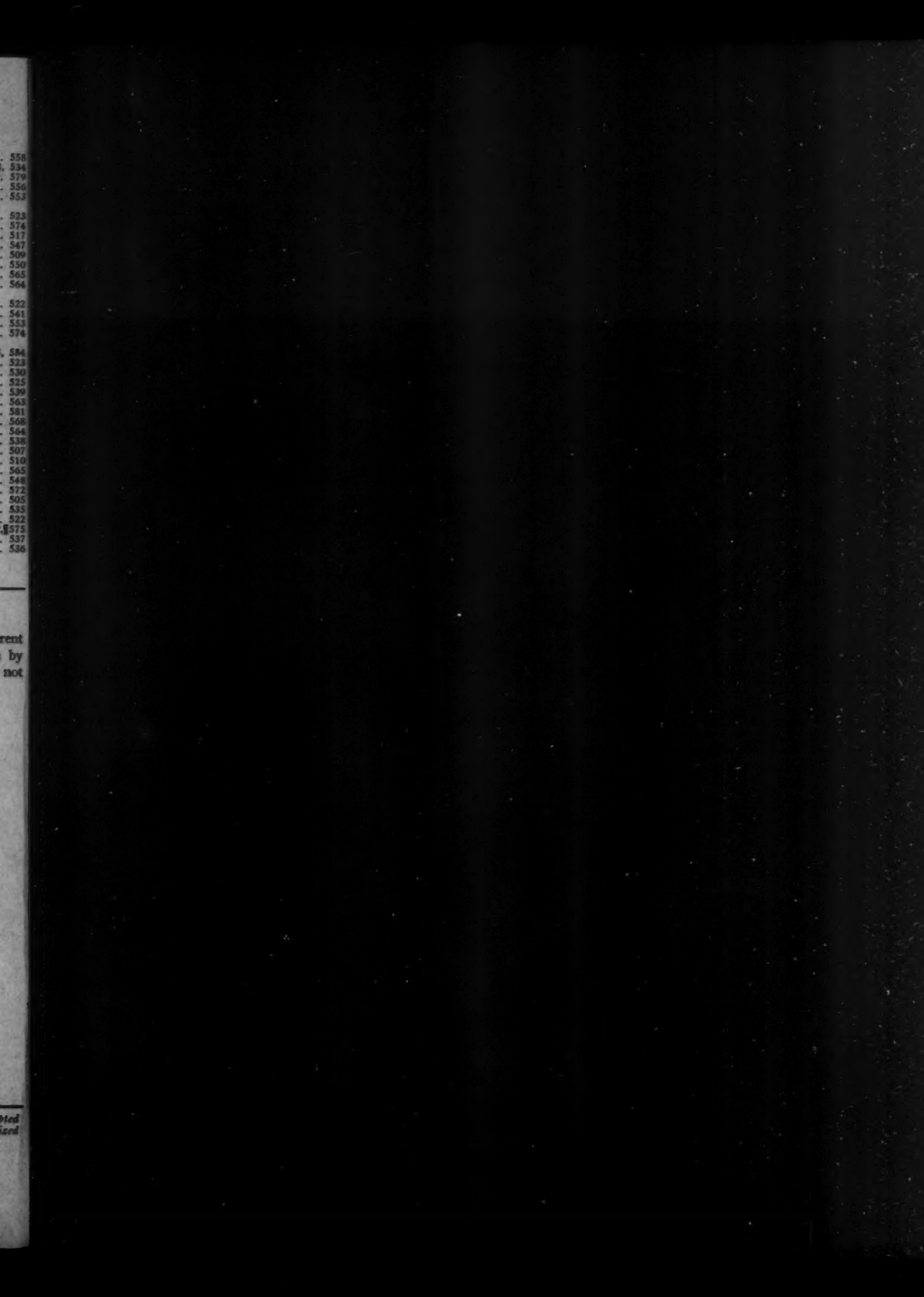

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FOUNDATIONS

Ridder, J. Über mehrwertige Aussagenkalküle und mehrwertige engere Prädikatenkalküle. III. Nederl. Akad. Wetensch., Proc. **51**, 991–995 = Indagationes Math. **10**, 324–328 (1948).

[For parts I and II cf. same Proc. **51**, 670–680, 836–845 = Indagationes Math. **10**, 221–231, 264–273 (1948); these Rev. **10**, 230.] A formulation is given of the Lewis system of strict implication $S1$, augmented by Becker's axiom $Np \rightarrow NNp$, in terms of disjunction, negation, assertion, and necessity (denoted by Np). There are nine axioms, one substitution rule, and two rules of inference. Two duality principles D_1 and D_2 are derived. The author shows that the system is free from contradiction by giving an interpretation with 2^n elements which are considered to be truth-values. This leads him to speak of the system as a many-valued logic with 2^n values. This system is called the system KL. It is pointed out that related systems denoted by $KL^{(p)}$ may be formed by taking direct products of several KL systems. *O. Frink* (State College, Pa.).

Ridder, J. Logic of propositions. Synthèse **6**, 496–502 (1948).

The author gives two equivalent formulations of the ordinary logic of elementary propositions, in terms of disjunction, negation, and assertion. The first of these systems has six axioms, one substitution rule, and one rule of inference. It is shown that the systems are free from contradiction by exhibiting an interpretation with 2^n elements. The author notes that the logic of predicates (restricted functional calculus) of Hilbert and Ackermann has likewise a 2^n-valued interpretation. He concludes that this shows the incorrectness of the commonly held view that the logic of classes is contained in the logic of predicates. The reviewer could not follow his reasoning here. *O. Frink.*

Brouwer, L. E. J. The non-equivalence of the constructive and the negative order relations in the continuum. Nederl. Akad. Wetensch., Proc. **52**, 122–124 = Indagationes Math. **11**, 37–39 (1949). (Dutch)

Every real number between 0 and 1 inclusive coincides with the limit of a sequence of intervals $(a_n 2^{-4n}, (a_n+2)2^{-4n})$, $n=1, 2, \cdots$, where each interval contains the next one. These sequences form a spread J. Let f be an element of J. We define $c_n(f)=0$ if at the moment of the choice of the nth interval $\lambda_n(f)$ of f neither the rationality nor the irrationality (absurdity of rationality) of f is known; if between the choices of $\lambda_r(f)$ and $\lambda_{r+1}(f)$ either the rationality or the irrationality of f becomes known, $c_{r+\nu}(f)=2^{-(r+\nu)}$ for $\nu=0, 1, \cdots$. The sequence of the $c_n(f)$ converges to a real number $d(f)$; let δ be the species of the $d(f)$ corresponding to the elements of J. Suppose now that the relations $x \mathrel{o>} 0$ (a positive rational number between 0 and x is known) and $x>0$ ($0 \mathrel{o>} x$ and $x=0$ are absurd) for real numbers were equivalent. For every element d of δ, $d>0$ is true, so $d \mathrel{o>} 0$ would be true, i.e., there corresponds to every d a natural number $n(d)$ such that $d \mathrel{o>} 2^{-n(d)}$; thus to every f in J there corresponds a natural number $n=n(d(f))$ such that after the choice of the nth interval of f, either the rationality or the irrationality of f is known. According to the fundamental theorem on spreads [Math. Ann. **97**, 60–75 (1926), p. 66] there is a maximal value N of n. As this leads immediately to a contradiction, the equivalence of the relations $\mathrel{o>}$ and $>$ between real numbers is proved contradictory. *A. Heyting* (Amsterdam).

Dequoy, Nicole. La géométrie projective plane en mathématique intuitioniste sans négation. C. R. Acad. Sci. Paris **228**, 1098–1100 (1949).

Assuming that it will be possible to formulate a theory of real numbers and an analytic geometry in a logic satisfying the demands of constructibility of Griss [Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **53**, 261–268 (1944); same C. R. **227**, 946–948 (1948); these Rev. **7**, 405; **10**, 277], the author proposes axioms for a projective geometry which it is hoped will be realized by this analytic geometry. The axioms concern points (elements of a set), straight lines (sets of points), the membership relation of point to line, and the relation of distinctness of points. Coincidence is defined in terms of the distinctness relation. Elementary theorems exhibiting the duality of point and line are observed, and it is stated that they may be derived in the formal logic of Destouches-Février [same C. R. **226**, 38–39 (1948); these Rev. **10**, 94]. An evaluation of the work will depend upon clarification of the principles of the negationless intuitionistic logic. *D. Nelson.*

Shukla, R. A system for general set theory. J. Indian Math. Soc. (N.S.) **12**, 121–124 (1948).

Consider the system of axioms which is obtained from Gödel's system Σ [cf. The Consistency of the Continuum Hypothesis, Princeton University Press, 1940; these Rev. **2**, 66] by deleting the axiom of infinity, replacing B 1 by '$(y)(\exists A)(x)(<xy>\varepsilon A . \equiv . x\varepsilon y)$' and B 5 by

'$(A)(\exists B)(x)(y)(<yx>\varepsilon B . \supset . x\varepsilon A)$'.

A nontrivial denumerable model is obtained for this system by interpreting classes as natural numbers written in dyadic notation and considering '$X\varepsilon Y$' meaningless unless Y contains more than X digits and true if and only if the Xth digit of Y is 1. The axiom of choice is not satisfied in this model, proving its independence from the other axioms. [Erratum: p. 122, line 8: '$\sim$Ex (A)' should read '$\sim$Em (A)'.] *I. L. Novak* (Wellesley, Mass.).

***Tarski, Alfred. A Decision Method for Elementary Algebra and Geometry.** The RAND Corporation, Santa Monica, Calif., 1948. iii+60 pp.

The system of elementary algebra can be described as follows. Signs: (1) a denumerable set of variables; (2) constants: 1, 0, -1, $+$, $\cdot$ (multiplication sign), $=$, $>$; (3) logi-

cal signs: $\sim$, $\wedge$, $\vee$, E (quantifier). Axioms: (i) classical lower predicate calculus with identity; (ii) a set of axioms for a commutative field with obvious properties for $1, 0, -1$; (iii) a continuity axiom to the effect that every rational integral function, which is positive at one point and negative at another, vanishes at some point between (as the author points out, this axiom can be replaced by simpler axioms). Though in this system the general notion of an integer is not definable (otherwise the result would contradict Gödel's theorem), it comprises important parts of mathematics. The axiomatic description of the system is given in a note; in the main text the author uses the idea of truth instead of that of derivability. In the following ϕ means any formula, ξ any variable, α and β are polynomials in ξ with arbitrary terms, not involving ξ, as coefficients. By simple transformations, ϕ is transformed into an equivalent formula which is a disjunction of conjunctions of atomic formulas $\alpha=0$ or $\alpha>0$. The derivative of a polynomial is defined. Then it is possible to express the following assertions by formulas: (1) ξ is a root of order n of α; (2) there exist exactly n values of ξ which make ϕ true; (3) there exists an open interval, whose right end-point is ξ, within which α and β have the same sign. Also the remainder obtained by dividing α by β (multiplied by a power of the leading coefficient in β to avoid fractions) is defined. Thus all the elements of a proof of Sturm's theorem are available. The decision method consists essentially in an extension of Sturm's theorem to an arbitrary system of equations and inequalities. More precisely, to the formula ϕ expressing that such a system has exactly k solutions, there is associated an equivalent formula $T\phi$ which contains no quantifiers and no variables except those that occur free in ϕ. Afterwards to any ϕ there is associated an equivalent formula $U\phi$ with the same properties. E.g., if $\phi=(\mathrm{E}\xi)\psi$, and $U\psi$ is already defined and equivalent to $\psi_1 \vee \cdots \vee \psi_n$, where each ψ_i is a conjunction of atomic formulas, then $U\phi=\sim T[\underset{0}{(\mathrm{E}\xi)}\psi_1] \vee \cdots \vee \sim T[\underset{0}{(\mathrm{E}\xi)}\psi_n]$, where $\underset{0}{(\mathrm{E}\xi)}\psi_i$ stands for "ψ_i has no solution in ξ." As any equation which contains neither variables nor quantifiers is equivalent to an elementary assertion in the arithmetic of integers, the decision method is now complete. Some extensions are discussed for which the method works as well as some for which the problem remains open. An example of the latter originates by the adjunction of exponentiation.

A. Heyting (Amsterdam).

***Szmielew, W. Decision problem in group theory.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 763–766 (1949).

This paper is concerned with the positive solution of the decision problem in the elementary (first order) theory of Abelian groups. (The solution of this problem for the elementary theory of arbitrary groups is known to be negative.) In terms of the satisfaction concept, it is shown that there exists a method which enables us to decide in each particular case whether a given sentence (of the theory under consideration) is satisfied by every Abelian group. Another form of this theorem is also arrived at by a different method. Some corollaries concerning the structural type of Abelian theory follow. *R. M. Martin* (Philadelphia, Pa.).

Tartakovskiĭ, V. On the problem of equivalence for certain types of groups. Doklady Akad. Nauk SSSR (N.S.) **58**, 1909–1910 (1947). (Russian)

This note continues investigations described in an earlier communication [same vol., 1605–1608 (1947); these Rev. **9**, 321]. For terminology and notation, see that review. The author computes a number $\delta>0$ from the words $f_1(S), f_2(S), \cdots, f_k(S)$, closely related to the maximum length of words obtainable by splitting the words $f_i(S)$ and forming products. The following theorem is stated without proof: if $\delta<\frac{1}{4}$, then an algorithm exists for determining in a finite number of steps whether or not two words of $\mathfrak{G}$ are equal. *E. Hewitt* (Seattle, Wash.).

Segre, Beniamino. Geometria dello spazio fisico. Archimede **1**, 73–81 (1949).

***van Dantzig, D. Blaise Pascal en de Betekenis der Wiskundige Denkwijze voor de Studie van de Menselijke Samenleving. [Blaise Pascal and the Significance of the Mathematical Way of Thought for the Study of Human Society].** P. Noordhoff N. V., Groningen, 1949. 37 pp.

ALGEBRA

***Nagell, Trygve. Lärobok i Algebra. [Textbook in Algebra].** Almqvist & Wiksells Akademiska Handböcker. Hugo Gebers Förlag. Uppsala, 1949. 303 pp.

The author states in the foreword: "The object of the present book is to give a brief introduction to the rather heterogeneous mathematical domain known as classical algebra. . . ." The chapter headings run as follows: (1) Polynomials in one or several variables, algebraic equations, number fields and rings, groupoids and groups; (2) Combinatorial analysis, arithmetic series; (3) Determinants, systems of linear equations; (4) Homogeneous linear substitutions, quadratic forms; (5) Symmetric functions, power sums and elementary symmetric functions, the principal theorem, applications of symmetric functions; (6) Distribution of roots of algebraic equations and numerical solution, approximation of roots, number of real roots in an interval; (7) Binomial and reciprocal equations, cubic equation, biquadratic equation, geometrical construction problems, elimination of terms in an algebraic equation; (8) Reducibility and irreducibility, relative algebraic numbers and fields, the solution of algebraic equations by radicals; (9) Abstract groups, permutation groups. The book is clearly written and a considerable number of examples add to the text. Among the topics treated in greater detail and with further results than those usually given may be mentioned: the reduction of the quintic equation to Bring's normal form, theorems on irreducibility, the cyclotomic equation and algebraic number fields. *O. Ore.*

Lévy, Paul. Sur deux classes de permutations. C. R. Acad. Sci. Paris **228**, 1089–1090 (1949).

The properties of classes of permutations considered in two preceding notes [cf. same C. R. **227**, 422–423, 578–579 (1948); these Rev. **10**, 177, 130] are unified and simplified. *J. Riordan* (New York, N. Y.).

Rohrbach, Hans. Die Anzahl der Zahlen mit vorgegebener Quersumme. Math. Nachr. **1**, 357–364 (1948).

The numbers concerned are those enumerating the number of numbers to base g ($g\geqq 2$) with at most n digits and a

sum of digits equal to q, a problem appearing in number codes. As noted by the author, these numbers have the generating function (due to de Moivre)

$$(1+x+\cdots+x^{q-1})^n=(1-x)^{-n}(1-x^q)^n$$

and are readily expressed as sums of products of binomial coefficients. For large values of n they are approximated by the normal distribution with mean $(q-1)n/2$ and variance $(q^2-1)n/12$. More precise estimates permitted by the central limit theorem are not considered. *J. Riordan.*

Smith, C. A. B., and Hartley, H. O. The construction of Youden squares. J. Roy. Statist. Soc. Ser. B. **10**, 262–263 (1948).

A symmetrical incomplete balanced block design is called a Youden square if every variety occurs once in every position. The authors show that every symmetrical balanced incomplete block design can be transformed into a Youden square by an iterative construction. This construction can be carried out efficiently in all cases that are at present of interest in the design of experiments. The proof does not use the condition that every pair of varieties occurs equally often, although the construction method does of course preserve this property. Thus the theorem proved is really more general. *H. B. Mann* (Columbus, Ohio).

Amato, Vincenzo. Sul gruppo di monodromia delle equazioni a gruppo algebrico G_n. Boll. Un. Mat. Ital. (3) **3**, 193–195 (1948).

The monodromy group of an equation whose Galois group is the normalizer of a permutation made up of ρ cycles of r elements ($\rho\neq 1$, $r\neq 1$) is known to be a transitive normal subgroup of the Galois group. It is shown that it is homomorphic to the symmetric group on ρ elements; in the case $r=\rho=2$ it is the Galois group itself, or is cyclic of size 4 or is a Vierer group. *H. A. Thurston* (Bristol).

van der Blij, F. A parametric representation for orthogonal matrices. Simon Stevin **26**, 74–80 (1949). (Dutch)

Cayley has given a parametric representation for orthogonal matrices A_n, which does not, however, hold for those for which $|A+E|=0$. The latter were considered by van der Woude [Nederl. Akad. Wetensch., Proc. **49**, 866–877 = Indagationes Math. **8**, 537–548 (1946); these Rev. **8**, 431] who by means of a limiting process obtained a representation for these exceptional cases. The author gives a generalization of these results. *O. Bottema* (Delft).

Parodi, Maurice. Remarque sur la stabilité. C. R. Acad. Sci. Paris **228**, 51–52 (1949).

It is shown that a condition sufficient for the characteristic equation $|A+zI|=0$ of a real matrix $A=(a_{ik})$ to have no roots z with positive real part is that $a_{ii}>0$ and $a_{ii}>\sum_{k\neq i}|a_{ik}|$, $i=1, 2, \cdots$. This follows from a well-known theorem on determinants. [For references see the review by G. B. Price of Massonnet, Bull. Soc. Roy. Sci. Liége **14**, 313–317 (1945); these Rev. **8**, 499.] *O. Todd-Taussky.*

Parodi, Maurice. Application d'un théorème de M. Hadamard à l'étude de la stabilité des systèmes. C. R. Acad. Sci. Paris **228**, 807–808 (1949).

Using a well-known theorem on determinants [cf. the preceding review], conditions are obtained sufficient to ensure that all the roots of the equation $|a_{ij}(z)|=0$ have negative real parts. Here a_{ij} ($j\neq i$) are real constants and $a_{ii}=(a_i+b_iz)/(\alpha_i+\beta_iz)$ with a_i, b_i, α_i, β_i real constants. *O. Todd-Taussky* (London).

Parodi, Maurice. Complément à un travail sur la stabilité. C. R. Acad. Sci. Paris **228**, 1198–1200 (1949).

The author [second preceding review] noted that if $a_{ii}>0$, $a_{ii}>\sum_{j\neq i}|a_{ij}|$, $i=1, \cdots, n$, then all the zeros of $|a_{ij}+\delta_{ij}z|$ have negative real parts and indeed lie in the domain D formed by the circles with centres $(-a_{ii}, 0)$ and radii $\sum_{j\neq i}|a_{ij}|$. He now considers the case when D touches the imaginary axis and finds a neighbourhood of the origin free from zeros by using a theorem of Ostrowski [Bull. Sci. Math. (2) **61**, 19–32 (1937)] which gives limits for $|\eta_{ij}|$ such that the matrices (a_{ij}) and $(a_{ij}+\eta_{ij})$ are both non-singular. *O. Todd-Taussky* (London).

Taussky, Olga. Bounds for characteristic roots of matrices. Duke Math. J. **15**, 1043–1044 (1948).

Let $A=(a_{ik})$ be an arbitrary square matrix of order n. It was proved by the reviewer [same J. **13**, 387–395 (1946); these Rev. **8**, 192] that each characteristic root of A lies in the interior or on the boundary of at least one of the n circles $|z-a_{\kappa\kappa}|\leqq\sum_{\lambda\neq\kappa}|a_{\lambda\kappa}|$ ($\kappa=1, 2, \cdots, n$). The author improves this result as follows. Assume that A cannot be reduced to the form

$$\begin{pmatrix} P & 0 \\ U & Q \end{pmatrix}$$

by the same permutation of the rows and columns, where P and Q are square matrices. A characteristic root of A can be a boundary point of the region formed by the n circles only if it is a common boundary point of all the circles. Moreover, the case $n=2$ is studied in greater detail. *A. Brauer* (Chapel Hill, N. C.).

Walker, A. G., and Weston, J. D. Inclusion theorems for the eigenvalues of a normal matrix. J. London Math. Soc. **24**, 28–31 (1949).

The following theorem is one of four proved. Let A be a normal matrix, x a vector, γ a number, and ρ the non-negative real number defined by

$$\rho^2=\left|\gamma-\frac{(Ax, x)}{\|x\|^2}\right|^2+\frac{\|Ax\|^2}{\|x\|^2}-\frac{|(Ax, x)|^2}{\|x\|^4}.$$

Then the circle with center γ and radius ρ contains an eigenvalue of A. The notation is $(x, y)=\sum x_i\bar{y}_i$ and $\|x\|^2=(x, x)$. The results given are shown to extend and sharpen L. Collatz's inclusion theorem on the eigenvalues of a real symmetric matrix of finite order [Math. Z. **48**, 221–226 (1942); these Rev. **5**, 30]. An extension to operators in Hilbert space is noted. *W. Givens* (Knoxville, Tenn.).

Turri, Tullio. A proposito degli automorfismi del corpo complesso. Rend. Sem. Fac. Sci. Univ. Cagliari **17** (1947), 88–94 (1948).

Die Behauptung von E. Steinitz, dass der Körper C der complexen Zahlen und der Körper aller algebraischen Funktionen isomorph sind [J. Reine Angew. Math. **137**, 167–309 (1910), Seite 301] führt Verfasser zu einem Widerspruch mit einem Ergebnis von B. Segre [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 414–420 (1947); diese Rev. **10**, 231] über die involutorischen Automorphismen von C, da diese durch genau zwei, aber nicht unendlich viele Automorphismen in ihren conjugierten transformiert werden können. Die Frage nach der Existenz von Automorphismen von C, die von der Identität und dem Übergang zur conjugiert complexen Zahl verschieden sind, ist äquivalent mit der Existenz von Punkttransformationen auf der complexen Geraden, die harmonische Quadrupel

erhalten und von den Homographien und Antihomographien verschieden sind. Verfasser vereinfacht seinen früheren Beweis für die Nichtexistenz solcher Transformationen [Rend. Sem. Fac. Sci. Univ. Cagliari **15**, 90–97 (1946); 216–222 (1947); diese Rev. **8**, 483; **9**, 54]. *R. Moufang.*

Abstract Algebra

Studley, Duane. Abstract isomorphism. Math. Mag. **22**, 191–193 (1949).

Riguet, J. Relations binaires, fermetures, correspondances de Galois. Bull. Soc. Math. France **76**, 114–155 (1948).

A binary relation is considered as a set of ordered pairs: $(x, y) \in R \subset E \times F$, also written xRy, with $x \in E$, $y \in F$. Then $yR^{-1}x$ means xRy; $xSRz$ means xRy and ySz; $R(x)$ denotes the set of y with xRy; $R(X)$ and $R[X]$ are the union and intersection of the $R(x)$ with $x \in X$. About 35 types of relations (including varieties of closure, functional, equivalence, cyclic, and order relations) are defined and their properties and interconnections analysed in detail. Example: provided the domains are properly related, $SR \cap T \subset (S \cap TR^{-1})(R \cap S^{-1}T)$ ("Dedekind relation"). The rectangular closure of R is the set of all (x, y) with $x \in R^{-1}(F)$, $y \in R(E)$. Say that R is difunctional if $RR^{-1}R = R$; R is an equivalence relation if it is reflexive and difunctional. For $S \subset E \times E$, E/S denotes the set of $S(x)$, $x \in E$. Theorem: if R is difunctional, then $R^{-1}R$ and RR^{-1} are equivalence relations on $R^{-1}(F)$ and $R(E)$, and there is a one-to-one mapping between $R^{-1}(F)/R^{-1}R$ and $R(E)/RR^{-1}$. If the domains are finite, a binary relation R can be represented by a matrix $\|a_{ij}\|$ with $a_{ij} = 1$ if $x_i R y_i$; otherwise $a_{ij} = 0$ [cf. Bull. Amer. Math. Soc. **48**, 174 (1942)].

Various types of bounds are defined in terms of order relations. Theorem: in a set E (with order relation Ω) which has greatest lower bounds, the set of all "C-closures" [$x \leqq C(x)$, $C(C(x)) = C(x)$, $x \leqq y$ implies $C(x) \leqq C(y)$] is dually isomorphic to the set of those meet-closed subsets of E such that $E = \Omega^{-1}(x)$. [Cf. Ore, Ann. of Math. (2) **44**, 514–533 (1943), especially theorem 4; these Rev. **5**, 170.] Galois correspondences [Ore, Trans. Amer. Math. Soc. **55**, 493–513 (1944); Everett, ibid., 514–525 (1944); these Rev. **6**, 36] are considered in the light of the previous work on relations and closures. *P. M. Whitman.*

Barbilian, D. Neue Gesichtspunkte über klassische Sätze aus der axiomatischen Algebra. Disquisit. Math. Phys. **6**, 3–48 (1948).

The greater part of the paper is devoted to a discussion of extensions of the second law of isomorphism in groups as well as the possible generalisations of the Jordan-Hölder and Schreier-Zassenhaus theorem for groups and later for lattices. The discussion is based upon a rather detailed study of various types of weak normality concepts, in particular the so-called α-normality introduced by the reviewer. A second part of the paper is concerned with the generalizations of the criterion of Dedekind-Hasse for principal ideal rings in algebraic number fields to more general types of rings satisfying chain conditions. Finally there is a brief discussion of Hilbert's basis theorem for the system of invariants in the case where the ring has a characteristic $p \neq 0$. *O. Ore* (New Haven, Conn.).

Barbilian, D. Zum Zerlegungsproblem der Algebra. Disquisit. Math. Phys. **6**, 243–262 (1948).

The present paper is characterized by the author as an appendix to that reviewed above. It contains a further examination of the various weak forms of normality in groups, in particular a type of "unit" normality and the corresponding form of the Schreier-Zassenhaus theorem. *O. Ore* (New Haven, Conn.).

Benado, Michaël. Nouveaux théorèmes de décomposition et d'intercalation attachés à la normalité α. C. R. Acad. Sci. Paris **228**, 529–531 (1949).

The author proves a Schreier-Zassenhaus theorem for lattices where the normality type is a so-called relative α-normality studied previously by Barbilian. *O. Ore.*

Mori, Shinjiro. Rings with the property of intersection decomposition and their idempotent ideals. J. Sci. Hirosima Univ. Ser. A. **12**, 205–215 (1943). (Japanese)

Mori, Shinjiro. On rings with the property of intersection decomposition. II. J. Sci. Hirosima Univ. Ser. A. **13**, 1–10 (1944). (Japanese)

In the first paper it is shown that, in a (commutative) ring with the property of intersection decomposition (that is, every ideal is decomposed into the intersection of a finite number of strongly primary ideals), (i) the ascending chain condition for prime ideals holds, and (ii) every idempotent ideal other than 0 contains an idempotent element other than 0. The former has the effect, when combined with a previous result of the author, that in order that a ring has the property of intersection decomposition it is necessary and sufficient that (i) holds and (iii) every chain of ideal quotients $\mathfrak{a} \subset \mathfrak{a} : b_1 \subset \mathfrak{a} : b_1 b_2 \subset \cdots$ is finite. In the second paper it is shown that if the ring has a unit element then (i) can be replaced by the weakened chain condition that (i′) every ascending chain of prime ideals contained in a fixed prime ideal (different from the ring itself) is finite. The author considers this as a preliminary work for his conjecture, left open in this paper, that (iii) alone will suffice. *T. Nakayama* (Nagoya).

Šul'gefer, E. G. The multiplicative theory of quasi-ideals in commutative rings. Doklady Akad. Nauk SSSR (N.S.) **64**, 633–636 (1949). (Russian)

This paper is a continuation of the study made by Andrunakievič of rings with respect to the operation $a+b-ab$ [Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 129–178 (1948); these Rev. **9**, 564]. The author is concerned with commutative rings (not necessarily semiradical) in which every quasi-ideal is uniquely expressible as a quasi-product of prime quasi-ideals. The following conditions are shown to be necessary and sufficient: the ascending chain condition, prime ideals maximal, all powers of a prime ideal different, no ideal between a prime ideal and its square (all taken in the "quasi" sense). This theorem and its proof are closely modelled on results for ordinary multiplication due to Moriya and Kobayasi [Proc. Imp. Acad. Tokyo **17**, 129–133, 134–138 (1941); these Rev. **3**, 101]. [Reviewer's remark: for ordinary multiplication the literature contains more general results not assuming uniqueness of decomposition, e.g., Mori, J. Sci. Hirosima Univ. Ser. A. **10**, 117–136 (1940); these Rev. **2**, 121.] *I. Kaplansky.*

Kawada, Yukiyosi. On the cohomology theory of rings. J. Math. Soc. Japan **1**, 22–28 (1948).

Following Hochschild [Ann. of Math. (2) **47**, 568–579 (1946); these Rev. **8**, 64] the cohomology groups $H^*(R, m)$ for a ring R with coefficients in a two-sided RR-module m are introduced. An RR-module M is an extension of m by the RR-module q if M, as a module, is the direct sum of

m and q and if $M/q \cong m$, as an RR-module. Each such extension may be described by a pair of factor sets which determine the left and right operations by elements of R; by means of these factor sets, the extension M determines a mapping F of $H^n(R, m)$ into $H^{n+1}(R, n)$. This mapping F is then used to obtain three results. (i) The Hochschild reduction theorem, $H^n(R, m) \cong H^{n+1}(R, q)$, when m is the module of all linear mappings of R into q; (ii) $H^1(R, n) = 0$ for every RR-module n if and only if every extension M of an m by a q is a direct sum, as an RR-module; (iii) $H^2(R, n) = 0$ for every n if and only if any extension M of m by q and any (suitable) ring extension A of R by m can be combined as an extension $B \supset M \supset q$ with $B/q \cong A$. The third result is an analogue of results of Teichmüller [Deutsche Math. **5**, 138–149 (1940); these Rev. **2**, 122]. *S. MacLane.*

Schwarz, Ludwig. Zur Theorie des nichtkommutativen Polynombereichs und Quotientenrings. Math. Ann. **120**, 275–296 (1948).

[A preliminary announcement of the results appeared in Ber. Math.-Tagung Tübingen 1946, pp. 134–136 (1947); these Rev. **9**, 173.] Let $\mathfrak{K}$ be a ring which possesses a 1-element. In the following, only subrings and extension rings of $\mathfrak{K}$ are considered which possess the same 1-element. It is assumed that there exist such subrings which are skew-fields. The author considers polynomials F in noncommutative indeterminates x_α with coefficients in $\mathfrak{K}$. For each of these x_α, a subring $\mathfrak{L}_\alpha$ of $\mathfrak{K}$ is supposed to be given such that x_α commutes with the elements of $\mathfrak{L}_\alpha$. Here $\mathfrak{L}_\alpha$ is assumed to be a skew-field. The polynomials F form the elements of an extension ring $\mathfrak{R}$ of $\mathfrak{K}$ which is generated by $\mathfrak{K}$ and the x_α and which has the following further properties: (a) If $c + \sum_\alpha \sum_i \mathfrak{c}_{\alpha i} x_\alpha F_{\alpha i} = c' + \sum_\alpha \sum_i \mathfrak{c}'_{\alpha i} x_\alpha F'_{\alpha i}$ with c, c', $\mathfrak{c}_{\alpha i}$, $\mathfrak{c}'_{\alpha i}$ in $\mathfrak{K}$ and $F_{\alpha i}$, $F'_{\alpha i}$ in $\mathfrak{R}$, then $c = c'$ and $\sum \mathfrak{c}_{\alpha i} x_\alpha F_{\alpha i} = \sum \mathfrak{c}'_{\alpha i} x_\alpha F'_{\alpha i}$. (b) If $\sum \mathfrak{a}_i x_\alpha F_i = 0$, where the F_i are in $\mathfrak{R}$ and where the $\mathfrak{a}_i$ are in $\mathfrak{K}$ and are right linearly independent with regard to $\mathfrak{L}_\alpha$, then all F_i are zero. The existence of such a ring $\mathfrak{R}$ is shown. If $\mathfrak{R}^*$ is another extension ring of $\mathfrak{K}$ which is generated by $\mathfrak{K}$ and a system of elements x_α^* such that x_α^* has $\mathfrak{L}_\alpha$ as its commuting ring, then there exists a homomorphic mapping of $\mathfrak{R}$ on $\mathfrak{R}^*$ which leaves every element of $\mathfrak{K}$ fixed and which maps x_α on x_α^*. A method is given by which it can be decided whether an element F of $\mathfrak{R}$ is zero. If $\mathfrak{K}$ has no divisors of zero, then $\mathfrak{R}$ has no divisors of zero. From now on, it is assumed that $\mathfrak{K}$ is a skew-field. If F and G are two elements of $\mathfrak{R}$ such that $F\mathfrak{R} \cap G\mathfrak{R} \neq 0$, a Euclidean algorithm can be given which yields a greatest common left divisor D such that $F\mathfrak{R} + G\mathfrak{R} = D\mathfrak{R}$. Now results can be established which are analogous to those given in N. Jacobson's "The Theory of Rings," pp. 29–34 [Amer. Math. Soc., New York, 1943; these Rev. **5**, 31] for general noncommutative principal ideal rings. [Under Jacobson's assumptions, it can be shown that $F\mathfrak{R} \cap G\mathfrak{R} \neq 0$ for any two nonzero elements of $\mathfrak{R}$.] In the last section, a full quotient ring $\mathfrak{Q}$ of $\mathfrak{R}$ is constructed. This is an extension ring $\mathfrak{Q}$ of $\mathfrak{R}$ such that every nonzero element of $\mathfrak{R}$ has an inverse in $\mathfrak{Q}$ and that $\mathfrak{Q}$ is generated by these inverses.

R. Brauer (Ann Arbor, Mich.).

Banerjee, D. P. On the self-inverse module. Math. Student **15** (1947), 17–18 (1948).

In a paper reviewed earlier [J. Indian Math. Soc. **3**, 295–306 (1939); these Rev. **1**, 198] F. W. Levi has investigated the structure of certain "inverse submodules" of a field. (Definitions, etc., may be found in this review.) In the present note a few additional properties of "self-inverse" modules are obtained. All are exercises upon, or elementary deductions from, Levi's results. A typical theorem is the following. If a, b belong to the self-inverse module A then $a^m b^n$ also belongs to A when $m+n$ is odd.

S. A. Jennings (Vancouver, B. C.).

Hasse, Helmut. Existenz und Mannigfaltigkeit abelscher Algebren mit vorgegebener Galoisgruppe über einem Teilkörper des Grundkörpers. II. Math. Nachr. **1**, 213–217 (1948).

A supplement to the investigations of the first part [Math. Nachr. **1**, 40–61 (1948); these Rev. **10**, 426] is given. In this earlier work of the author, two systems of equations were of fundamental importance. These systems of equations are now interpreted as statements concerning isomorphisms of groups. *R. Brauer* (Ann Arbor, Mich.).

Hasse, Helmut. Existenz und Mannigfaltigkeit abelscher Algebren mit vorgegebener Galoisgruppe über einem Teilkörper des Grundkörpers. III. Math. Nachr. **1**, 277–283 (1948).

This paper contains some further supplements to the earlier investigations of the author [cf. the preceding review]. Let K/Ω be an Abelian algebra with the group $\mathfrak{A}$. Let K_0, Ω_0, Γ, $\mathfrak{G}$, $\mathfrak{A}$ have the same significance as in the first part. Then the following question is raised: In what manner is the fact that K/Ω_0 is a Galois algebra with the group $\mathfrak{G}$ expressed in the structure of K_0/Ω_0? The answer is as follows. If K_0/Ω belongs to the subgroup $\mathfrak{A}_0$ of $\mathfrak{A}$, then: (1) the representation Γ transforms $\mathfrak{A}_0$ into itself and, consequently, Γ defines a representation Γ_0 by means of automorphisms of $\mathfrak{A}_0$; (2) the factor set C defining $\mathfrak{G}$ is associated to a factor set C_0 consisting of elements of $\mathfrak{A}_0$. Then the Galois group $\mathfrak{G}_0$ of K_0/Ω_0 is the extension group of $\mathfrak{A}_0$ which is defined by means of the representation Γ_0 and the class of associated factor sets containing C_0. In the second section, the conjecture mentioned in the first part is proved in certain special cases. *R. Brauer* (Ann Arbor, Mich.).

Rozenfel'd, B. A. The geometry of simple algebras. Doklady Akad. Nauk SSSR (N.S.) **64**, 629–632 (1949). (Russian)

Consider an algebra A over the real numbers, having a unit element, and with an involution such that precisely the self-adjoint elements are real; A is either a division algebra, a 2 by 2 matrix algebra, or the direct sum of two fields isomorphic to the real numbers. The author surveys the topological spaces and groups associated with these algebras. *I. Kaplansky* (Chicago, Ill.).

Barsotti, I. Osservazioni elementari intorno al differente di un corpo o di un'algebra sopra un corpo algebrico, o sopra un corpo di funzioni algebriche di una variabile. Boll. Un. Mat. Ital. (3) **3**, 223–227 (1948).

(1) Let $\mathfrak{o}$ be the integers of an algebraic number field F, $\mathfrak{O}$ a maximal order in a central simple algebra over F; or (2) let $\mathfrak{o}$ be the polynomial ring in a transcendental extension $k(x)$ of a field k of characteristic 0, $\mathfrak{O}$ the integral elements in a finite algebraic extension of $k(x)$. In either case suppose a prime ideal $\mathfrak{p}$ in $\mathfrak{o}$ factors $\mathfrak{p} = P^e \cdots$ in $\mathfrak{O}$. Then the relative different contains P precisely to the $(e-1)$th power. The proofs are brief and elementary.

I. Kaplansky (Chicago, Ill.).

Albert, A. A. On right alternative algebras. Ann. of Math. (2) **50**, 318–328 (1949).

The author studies nonassociative algebras in which the identity $y(xx) = (yx)x$ holds, and calls them right alterna-

tive. They include the alternative algebras investigated by Zorn [Abh. Math. Sem. Hamburg. Univ. **8**, 123–147 (1930); **9**, 395–402 (1933); same Ann. (2) **42**, 676–686 (1941); these Rev. **3**, 100]. Over a field F of characteristic not two, every right alternative algebra $\mathfrak{A}$ is power-associative. If $\mathfrak{A}$ is of degree two, then $\mathfrak{A}$ is alternative. The Peirce decomposition relative to an idempotent does not give results analogous to the alternative case (where this decomposition leads to a structure theory over an arbitrary field). For this reason the author employs tools he introduced in a recent paper on power-associative rings [Trans. Amer. Math. Soc. **64**, 552–593 (1948); these Rev. **10**, 349]. A Jordan algebra $\mathfrak{A}^{(+)}$ is obtained by defining a new multiplication $x \cdot y = \frac{1}{2}(xy+yx)$ in $\mathfrak{A}$. The mapping $x \to R_x$ is a homomorphism of $\mathfrak{A}^{(+)}$ onto the Jordan algebra $R(\mathfrak{A})$ of right multiplications of $\mathfrak{A}$.

Assume that F has characteristic zero. Then the author's results on the structure of Jordan algebras [same Ann. (2) **48**, 546–567 (1947); these Rev. **9**, 77] may be applied. Also the trace $\tau(L_{xy})$ of the left multiplication L_{xy} is an admissible trace function $\tau(x, y)$ for $\mathfrak{A}$. The maximal nilideal $\mathfrak{N}$ of $\mathfrak{A}$ (the set of all x such that $\tau(x, y) = 0$ for every y) coincides with the radical of $\mathfrak{A}^{(+)}$. The algebra $\mathfrak{A}$ is called semisimple in case $\mathfrak{N} = 0$. The author's chief result is that, if $\mathfrak{A}$ is semisimple over F (of characteristic zero), then $\mathfrak{A}$ is alternative. Consequently every simple right alternative algebra over F is either associative or a Cayley algebra over its center. *R. D. Schafer* (Philadelphia, Pa.).

Harish-Chandra. Faithful representations of Lie algebras. Ann. of Math. (2) **50**, 68–76 (1949).

This paper contains a proof of the theorem that every Lie algebra $\mathfrak{L}$ over a field K of characteristic zero has a faithful representation. The first proof of this result, at least for the case in which K is algebraically closed, was given by Ado [Bull. Soc. Phys.-Math. Kazan (3) **7**, 3–43 (1936)]. The present proof is purely algebraic and approaches the problem quite directly. Moreover, it is shown that a faithful representation of $\mathfrak{L}$ can be so chosen that every element of the maximal nilpotent ideal of $\mathfrak{L}$ is mapped into a nilpotent matrix. Once the theorem is established for solvable Lie algebras $\mathfrak{L}$, it follows quickly for arbitrary $\mathfrak{L}$ with the aid of the theorem of Levi. For solvable $\mathfrak{L}$ the theorem is proved by an induction on the number of dimensions. The key is furnished by a lemma which asserts the existence of a faithful representation (satisfying certain conditions concerning nilpotency) of $\mathfrak{L}+\mathfrak{D}$, where $\mathfrak{L}$ is any solvable linear Lie algebra, $\mathfrak{D}$ the algebra of its derivations, and $\mathfrak{L}+\mathfrak{D}$ the semidirect sum of $\mathfrak{L}$ and $\mathfrak{D}$, that is, the totality of pairs (X, D), $X \in \mathfrak{L}$, $D \in \mathfrak{D}$, with bracket operation $[(X, D), (X', D')] = ([X, X']DX' - D'X, [D, D])$.

P. A. Smith (New York, N. Y.).

THEORY OF GROUPS

Robinson, G. de B. On the disjoint product of irreducible representations of the symmetric group. Canadian J. Math. **1**, 166–175 (1949).

Given n sets each of m symbols, H denotes the direct product of the symmetric groups on each set. If $[\alpha], [\beta], \cdots, [\gamma]$ denote matrix representations of these symmetric groups then the direct product $[\alpha]\cdot[\beta]\cdot\cdots\cdot[\gamma]$ gives a representation of H. If the representations are all the same so that $[\alpha]=[\beta]=\cdots=[\gamma]$ then $\Pi_n^{(\alpha)} = [\alpha]\cdot[\alpha]\cdot\cdots\cdot[\alpha]$ gives a representation which may be extended to the imprimitive group $N(H)$ obtained from H by allowing a symmetric group of permutations on the sets among themselves. The representation is then reducible according to the various representations of the symmetric group on the sets according to the formula $\Pi_n^{(\alpha)} = \sum_\beta \chi_\beta [\alpha] \odot [\beta]$. On further extension to the symmetric group on mn symbols the representation $[\alpha]\odot[\beta]$ is in exact correspondence with the reviewer's "new multiplication" or "plethysm" of S-functions $\{\alpha\}\otimes\{\beta\}$ and this has considerable application to invariant theory.

If $[\alpha]\odot[\beta]$ includes the representation $[\lambda]$ of the symmetric group on mn symbols f_λ times the author obtains the formula for f_λ:

$$f_\lambda = \frac{1}{(m!)^n n!} \sum \chi_\delta(s)[\chi(hs)\cdot\varphi_\lambda(hs)].$$

The result gives a method of expanding $\{\alpha\}\otimes\{\beta\}$. It must be remarked, however, that though the result has intrinsic interest, the amount of calculation required and the use of character tables required would seem to prevent the method from having important practical use in invariant theory. The reviewer's theorem of conjugates follows immediately from the results obtained.

D. E. Littlewood (Bangor).

Grün, Otto. Beiträge zur Gruppentheorie. II. Über einen Satz von Frobenius. J. Reine Angew. Math. **186**, 165–169 (1945).

[Part I appeared in J. Reine Angew. Math. **174**, 1–14 (1935), and part III is reviewed below.] The theorem of Frobenius in question is the following. If the finite group $\mathfrak{G}$ contains a subgroup $\mathfrak{U}$ which is its own normaliser, and which has only the identity in common with any of its conjugates in $\mathfrak{G}$, then the elements of $\mathfrak{G}$ not lying in $\mathfrak{U}$ or any of its conjugates form, together with the identity, a normal subgroup $\mathfrak{H}$ of $\mathfrak{G}$ such that $\mathfrak{G} = \mathfrak{H}\mathfrak{U}$, $\mathfrak{G}/\mathfrak{H} \cong \mathfrak{U}$ [cf. Speiser, Die Theorie der Gruppen von endlicher Ordnung, 3d ed., Springer, Berlin, 1937, p. 202, theorem 180]. The author states that this theorem has so far been proved only by methods involving the use of group characters, and shows that if the following lemma could be obtained without them a group-theoretical proof of the above theorem would follow. Lemma: if $\mathfrak{U}$ satisfies the hypotheses of the above theorem, there exists a system of right (left) coset representatives of $\mathfrak{U}$ in $\mathfrak{G}$ which are simultaneously right (left) representatives for all the conjugates of $\mathfrak{U}$ in $\mathfrak{G}$. A group-theoretical proof is given of the following modification of Frobenius's theorem. Let $\mathfrak{U}$ be a subgroup of $\mathfrak{G}$ which is its own normaliser, and let the intersection of all the conjugates of $\mathfrak{U}$ in $\mathfrak{G}$ be a proper subgroup $\mathfrak{U}_1$ of $\mathfrak{U}$. Moreover let there exist a proper normal subgroup $\mathfrak{U}_2 \supseteq \mathfrak{U}_1$ of $\mathfrak{U}$ so that $\mathfrak{U}/\mathfrak{U}_2$ is solvable. Then there exists a normal subgroup $\mathfrak{H}$ of $\mathfrak{G}$ such that $\mathfrak{H}\mathfrak{U} = \mathfrak{G}$, $\mathfrak{H} \cap \mathfrak{U} = \mathfrak{U}_2$, with $\mathfrak{G}/\mathfrak{H} \cong \mathfrak{U}/\mathfrak{U}_2$. The proof uses ideas introduced by the author in part I.

S. A. Jennings (Vancouver, B. C.).

Grün, Otto. Beiträge zur Gruppentheorie. III. Math. Nachr. **1**, 1–24 (1948).

[Cf. the preceding review.] In what follows $\mathfrak{G}$ is a finite group, $\mathfrak{U}$ a subgroup of $\mathfrak{G}$, $\mathfrak{Z}(\mathfrak{U})$ denotes the center of $\mathfrak{U}$, $\mathfrak{N}(\mathfrak{U})$ the normaliser of $\mathfrak{U}$ in $\mathfrak{G}$, and $\mathfrak{P}$ is a p-Sylow sub-

group of $\mathfrak{G}$. The first part of this paper is devoted to a consideration of how two elements, complexes or subgroups of $\mathfrak{P}$ are conjugate in $\mathfrak{G}$. Illustrative of the numerous theorems proved in this connection is the following, a generalization of a result obtained by the author in part I of this study [reference in the preceding review]. Let $\mathfrak{Z}(\mathfrak{P}), \mathfrak{Z}(\mathfrak{P}^{s_1}), \mathfrak{Z}(\mathfrak{P}^{s_2}), \cdots, \mathfrak{Z}(\mathfrak{P}^{s_n})$ be the conjugates of $\mathfrak{Z}(\mathfrak{P})$ in $\mathfrak{G}$ which lie in $\mathfrak{P}$; then two subgroups of $\mathfrak{P}$ (or more generally two subgroups or complexes of $\mathfrak{G}$ which are invariant under $\mathfrak{Z}(\mathfrak{P})$) are conjugate under an element of the complex

$$K(\mathfrak{P})=\mathfrak{N}(\mathfrak{Z}(\mathfrak{P}))(1+S_1+\cdots+S_n)\times(1+S_1^{-1}+\cdots+S_n^{-1})\mathfrak{N}(\mathfrak{Z}(\mathfrak{P}))$$

or are not conjugate in $\mathfrak{G}$. The complex $K(\mathfrak{P})$ is called the "conjugating complex" of $\mathfrak{P}$; it is determined only by the situation of $\mathfrak{Z}(\mathfrak{P})$ in $\mathfrak{G}$ and plays an important role in the theory developed by the author.

The second part of the paper is concerned with intersections of conjugate subgroups, and especially Sylow subgroups. Let $\mathfrak{U}_1, \cdots, \mathfrak{U}_s$ be any set of subgroups of $\mathfrak{G}$, none being contained in any of the others; form all intersections of the $\mathfrak{U}_1, \cdots, \mathfrak{U}_s$ taken $1, 2, \cdots$ at a time. If $\mathfrak{D}$ is an intersection of the $\mathfrak{U}$ we define $\mathfrak{D}$ to be of order 1 if it is not properly contained in any other intersection, and write $\mathfrak{D}=\mathfrak{D}_1$. By induction an intersection $\mathfrak{D}$ is defined to be of order k if it is properly contained only in intersections of orders not exceeding $k-1$, and we write $\mathfrak{D}=\mathfrak{D}_k$. If the $\mathfrak{U}_1, \cdots, \mathfrak{U}_s$ consist of a subgroup $\mathfrak{U}$ and its conjugates in $\mathfrak{G}$, the set of all intersections is called the $\mathfrak{U}$-intersection system of $\mathfrak{G}$. The author obtains many results connecting $\mathfrak{U}$-intersection systems with other conjugate sets in $\mathfrak{G}$. For example, if $\mathfrak{W}$ is a normal subgroup of $\mathfrak{U}$ which is also contained in a conjugate of $\mathfrak{U}$, then there exists a normal subgroup of $\mathfrak{U}$ containing $\mathfrak{W}$ which belongs to the $\mathfrak{U}$-intersection system. If $\mathfrak{U}, \mathfrak{U}^s, \mathfrak{U}^r, \cdots$ are those conjugates of $\mathfrak{U}$ in $\mathfrak{G}$ which have elements not equal to 1 in common with $\mathfrak{U}$, and if $\mathfrak{D}$ is the intersection of these subgroups, then $\mathfrak{N}(\mathfrak{U}), \mathfrak{N}(\mathfrak{U}^s), \mathfrak{N}(\mathfrak{U}^r), \cdots$ lie in $\mathfrak{N}(\mathfrak{D})$ and are conjugate in $\mathfrak{N}(\mathfrak{D})$. The case of a $\mathfrak{P}$-intersection system is discussed at length. For example, it is shown that if $\mathfrak{D}_k$ is a $\mathfrak{P}$-intersection of order $k>1$, then there exists a $\mathfrak{P}$-intersection $\mathfrak{D}_1$ such that $\mathfrak{P}\supset\mathfrak{D}_1\supset\mathfrak{D}_k$. Generalizations of Zassenhaus's theorem on first order $\mathfrak{P}$-intersections are obtained [cf. Zassenhaus, Lehrbuch der Gruppentheorie, v. 1, Teubner, Leipzig and Berlin, 1937, especially theorem 7, page 102] and relations between the subgroups of the lower central series of a given $\mathfrak{P}$-intersection $\mathfrak{D}$ and the corresponding subgroups of the Sylow groups containing $\mathfrak{D}$ are indicated.

Finally generalizations of the notion of a p-normal group introduced by the author in part I are given. Let $\mathfrak{B}(\mathfrak{P})$ be the weak closure of $\mathfrak{Z}(\mathfrak{P})$ in $\mathfrak{P}$ with respect to $\mathfrak{G}$ [cf. Wielandt, J. Reine Angew. Math. **182**, 180–193 (1940); these Rev. **2**, 216]. Then $\mathfrak{G}$ is called p-regular if $\mathfrak{B}(\mathfrak{P})$ is contained in every group which contains $\mathfrak{Z}(\mathfrak{P})$; and $\mathfrak{G}$ is called p-hyper-regular if, for every $\mathfrak{P}$-intersection $\mathfrak{D}$, there exists a p-Sylow subgroup of $\mathfrak{N}(\mathfrak{D})$ which is in $\mathfrak{P}$ whenever $\mathfrak{D}$ is in $\mathfrak{P}$. Every hyper-regular group is regular, but it is an open question whether there are p-regular groups which are not p-hyper-regular. For a p-regular group $\mathfrak{G}$ the relation $K(\mathfrak{P})=\mathfrak{N}(\mathfrak{B}(\mathfrak{P}))$ holds, and the maximal Abelian p-factor group of $\mathfrak{G}$ is isomorphic with that of $\mathfrak{N}(\mathfrak{B}(\mathfrak{P}))$. Indeed, the author proves the more general result that, for any group $\mathfrak{G}$, this factor group is isomorphic with that of any subgroup $\mathfrak{U}$ of $\mathfrak{G}$ which contains the conjugating complex $K(\mathfrak{P})$ of $\mathfrak{G}$. *S. A. Jennings* (Vancouver, B. C.).

Szele, T. Über die direkten Teiler der endlichen Abelschen Gruppen. Comment. Math. Helv. **22**, 117–124 (1949).

A finite Abelian p-group will be called regular if its invariants are all equal, that is, if it is of type $(p^n, \cdots, p^n)$. If G is any finite Abelian group, G may be written as a direct product of Abelian regular p-groups, $G=G_1\times G_2\times\cdots\times G_l$, where G_i is of type $(p_i^{n_i}, \cdots, p_i^{n_i})$ and order $p_i^{s_i n_i}$, say. The author proves that a subgroup H of G is a direct factor of G if and only if H may be written in the form $H=H_1\times H_2\times\cdots\times H_l$, where H_i is a (not necessarily proper) subgroup of G_i of type $(p_i^{n_i}, \cdots, p_i^{n_i})$ and order $p_i^{m_i n_i}$, with $0\leqq m_i\leqq n_i$ for all i. The proof reduces at once to the case where G is of prime power order and a criterion for a subgroup H of such a group G to be a direct factor of G is obtained in terms of the coefficient matrix in the expression for a basis of H relative to a basis of G. As the author himself remarks in an addendum, the proofs could be very considerably shortened by making use of earlier results along the same lines due to Prüfer [Math. Z. **17**, 35–61 (1923)] and Shoda [Math. Ann. **100**, 674–686 (1928)].

S. A. Jennings (Vancouver, B. C.).

Kaloujnine, Léo. La structure des p-groupes de Sylow des groupes symétriques finis. Ann. Sci. École Norm. Sup. (3) **65**, 239–276 (1948).

Let E_m be the set of all vectors $X=(x_1, \cdots, x_m)$ with components in $GF(p)$. There are p^m such vectors X, and any one-to-one mapping of E_m onto itself may be considered as a permutation of the symmetric group $\mathfrak{S}_{p^m}$ of degree p^m. Now any mapping of E_m into itself is of the form $X\to Y=(y_1, \cdots, y_m)$ for all X in E_m, and we may write $y_i=f_i(x_1, \cdots, x_m)$, $i=1, \cdots, m$, where because of the finiteness of the underlying field and the fact that $x^p=x$ for $x\in GF(p)$, the functions f_i may be taken to be polynomials in the x_i reduced modulo the ideal $(x_1^p-x_1, \cdots, x_m^p-x_m)$. In particular the group $\mathfrak{S}_{p^m}$ may be realized as a substitution group of polynomials of the form $y_i=f_i(x_1, \cdots, x_m)$, $i=1, \cdots, m$, where all f_i which appear are reduced so that they are of degrees at most $p-1$ in each component. Some mention of this type of group (and the natural generalisation to $GF(p, n)$) may be found in L. E. Dickson's "Linear Groups with an Exposition of the Galois Field Theory" [Teubner, Leipzig, 1901] but no systematic discussion appears to have been given. The author's starting point in the present paper is the remark that a p-Sylow subgroup of $\mathfrak{S}_{p^m}$ may be obtained by all substitutions of the form $y_1=x_1+a$, $y_2=x_2+a_2(x_2)$, $\cdots$, $y_m=x_m+a_m(x_1, \cdots, x_{m-1})$, where $a\in GF(p)$ and a_k is a polynomial in $x_1, \cdots, x_{k-1}$, $k=2, 3, \cdots, m$. He then investigates some of the properties of this Sylow group $\mathfrak{P}$ by means of the above realization. All characteristic subgroups of $\mathfrak{P}$ are specified algebraically, and the commutator structure of $\mathfrak{P}$ is determined, using the fact that commutation lowers the degrees of certain of the polynomials a_k. Since almost all the results of the present paper have been announced previously [C. R. Acad. Sci. Paris **221**, 222–224 (1945); **222**, 1424–1425 (1946); **223**, 703–705 (1946); **224**, 253–255 (1947); these Rev. **7**, 239; **8**, 13, 251, 367] reference should be made to the earlier reviews for details. *S. A. Jennings* (Vancouver, B. C.).

Gol'berg, P. A. Sylow bases of π-separable groups. Doklady Akad. Nauk SSSR (N.S.) **64**, 615–618 (1949). (Russian)

Let σ_k be a collection of k primes. If every index in a composition series of a finite group G is divisible by at most

one prime of σ_k, then G is called π-separable (σ_k), or π-separable. If σ_k contains all the prime factors of the order of the group, then the group is soluble. Let Σ be a set of Sylow subgroups of G; suppose that, for each subset T of Sylow subgroups from Σ, the order of the group generated by T is divisible by no primes other than those which correspond to the Sylow groups in T. Then Σ is called a Sylow basis in G. Theorem 1. Let a finite group G be π-separable (σ_k). Then G has a Sylow basis which contains one Sylow subgroup corresponding to each prime in σ_k. Proof: If $k=1$, this is Sylow's theorem. If $k>1$, there is a normal subgroup H, $G \supset H \supset 1$; suppose p_1 $[\epsilon\sigma_k]$ divides the index of H. By induction hypothesis, H has a Sylow basis $\{S_1', S_2, \cdots, S_k\}$, where $S_2, \cdots, S_k$ are Sylow subgroups of G belonging to $p_2, \cdots, p_k$. Let $S(G)$ $[S(H)]$ be the set of those elements in G $[H]$ which transform this Sylow basis into itself. The indices of $S(G)$ $[S(H)]$ in G $[H]$ are equal; from this it follows that $\{S_1'P_1, S_2, \cdots, S_k\}$ is a Sylow basis in G, where P_1 is the Sylow subgroup of $S(G)$ corresponding to the prime p_1. Theorem 2. In G, two Sylow bases corresponding to the same set of primes are mutually conjugate. These theorems generalize results of P. Hall [Proc. London Math. Soc. (2) **43**, 316–323 (1937)] and S. A. Čunihin [Doklady Akad. Nauk SSSR (N.S.) **59**, 443–445 (1948); these Rev. **9**, 492]. *J. L. Brenner.*

Hall, Marshall, Jr. Subgroups of finite index in free groups. Canadian J. Math. **1**, 187–190 (1949).

The principal result of this paper is the proof of the following recursion formula for the number $N(n, r)$ of subgroups of index n of a free group on r generators (where n and r are finite positive integers and where $N(1, r)=1$): $N(n, r)=n(n!)^{r-1}-\sum_{i=1}^{n-1}[(n-i)!]^{r-1}N(i, r)$. The argument is based on the theory of Schreier systems [Hall and Radó, Trans. Amer. Math. Soc. **64**, 386–408 (1948); these Rev. **10**, 98]. *R. Baer* (Urbana, Ill.).

Baer, Reinhold. Groups with descending chain condition for normal subgroups. Duke Math. J. **16**, 1–22 (1949).

Let G be a group the set of whose normal subgroups satisfies the descending chain condition. The author seeks to prove finiteness of suitable parts of G, and limitations on the lengths of suitable ascending chains of normal subgroups of G. Let $S_F(G)=S_F(G, 1)$ be the subgroup generated by the finite minimal normal subgroups of G; and define $S_F(G, \lambda)$ for all ordinals λ by (i) $S_F(G, \mu+1)/S_F(G, \mu)=S_F(G/S_F(G, \mu))$, (ii) if ρ is a limit ordinal $S_F(G, \rho)$ is the union of $S_F(G, \mu)$ for $\mu<\rho$. Then this series terminates after less than $\omega \cdot 2$ terms; $S_F(G, \omega)$ is of finite index in $S_F(G, \omega \cdot 2)$ and consists of those elements which have only a finite number of conjugates in G; and $S_F(G, \omega)$ contains as a subgroup of finite index a normal subgroup A of G which is a direct sum of a finite number of Abelian groups of type p^∞, for various primes p. In particular, if G has a principal series in which every factor group is finite, so that $G=S_F(G, \lambda)$ for some λ, then G has a normal subgroup of this kind which is of finite index in G; and the converse is also true. It also follows that the ascending central series of G terminates after less than $\omega \cdot 2$ terms. Finally, if $J(G)$ is the intersection of the subgroups of G of finite index, we have the commutator identities $[G, [J(G), J(G)]]=[J(G), J(G)]$ and $[G, [G, J(G)]]=[G, J(G)]$. *G. Higman* (Manchester).

Baer, Reinhold. Extension types of Abelian groups. Amer. J. Math. **71**, 461–490 (1949).

All groups considered are Abelian. Two extensions G and H of a group S [i.e., groups containing S as subgroup] are called equivalent if there exists between them an isomorphism leaving S element-wise invariant; and they are of the same extension type if there are homomorphisms of G into H and of H into G leaving S element-wise invariant. Partial ordering can be introduced into the set of extension types of a given group S by defining that, of two extensions G and H of S, the extension G has the smaller type if G can be mapped homomorphically into H leaving S element-wise invariant.

The author proposes the problem of an internal characterisation of the set of all extension types of a given group S and gives a complete solution for the restricted case of little extensions. A little extension G of S is one for which G/S has no elements of infinite order, and no elements of order p (prime) if G has an infinity of elements of order p. In the case of little extensions the set of extension types can be characterized by means of Loewy chains. A Loewy chain of S for the prime p is a sequence of subgroups $S=S_{(0)} \supseteq S_{(1)} \supseteq S_{(2)} \supseteq \cdots$ with the property $S_{(i+1)} \supseteq pS_{(i)}$ so that $S_{(i)}/S_{(i+1)}$ is a direct sum of cyclic groups of order p. The subgroups p^iS, $i=0, 1, 2, \cdots$, always form a p-Loewy chain: the trivial one. It is easy to see that any little extension G of S gives rise to a p-Loewy chain for S with the terms $S(i; p)=S \cap p^iG$ for every prime power p^i. If this is, for some p, not the trivial Loewy chain then it has the following properties: (i) S contains only a finite number of elements of order p; (ii) $[pS(i, p) \cap S(i+2; p)]/pS(i+1; p)$ is finite for all i and differs from zero for only a finite number of i. The main result of the paper is now that these necessary conditions on nontrivial Loewy chains are also sufficient for the existence of an extension G of S with $S(i; p)=S \cap p^iG$ as terms of its p-Loewy chains. The construction, by induction, of an extension with given invariants is rather intricate.

A Loewy system $L=[L(2), L(3), \cdots, L(p), \cdots]$ of S consists of a set of definite Loewy chains for each prime p. Partial ordering can be introduced into the set of all Loewy systems of S by defining that of two Loewy systems L and $L'=[L'(2), L'(3), \cdots, L'(p), \cdots]$, $L \leqq L'$ if $L(p) \leqq L'(p)$ for every prime p, the latter notation meaning that $S(i; p) \leqq S'(i; p)$, $i=0, 1, 2, \cdots$, for the associated subgroups of S. A Loewy system L is called little if it satisfies conditions (i) and (ii) above for all those primes p for which the corresponding p-Loewy chain $L(p)$ is not the trivial one. It is then shown that a (natural) one-to-one correspondence exists between little extensions of S and little Loewy systems of S so that the partially ordered set $\Lambda(s)$ of little extensions of S can be characterized internally by the partially ordered set $L(s)$ of little Loewy systems of S.

If G is an extension of S and T is arbitrary, then the direct sum $G \oplus T$ is an extension of the same type as G (without, in general, being equivalent to G). A little extension G of S is called minimal if no proper direct summand of G is an extension of direct decomposition $G=G' \oplus G''$ where G' is a minimal extension. Minimal extensions belonging to the same type are equivalent, and within a given type of extensions equivalence and isomorphism coincide. In the final part of the paper restrictions are imposed upon S itself to yield more detailed information. The group S is called a little group if every Loewy system of S is little. (Free Abelian groups are not little, but finite groups are.) For a little group the set of little extension types is a complete modular lattice.

K. A. Hirsch (Newcastle-upon-Tyne).

Vilenkin, N. Ya. On a method of constructing topological groups. Doklady Akad. Nauk SSSR (N.S.) **65**, 3–5 (1949). (Russian)

Suppose given a fixed system of topological groups: $G_a, G_b, \cdots, G_w, \cdots$, the subscripts being drawn from a set M. Following a procedure due to Köthe and Toeplitz [J. Reine Angew. Math. **171**, 193–226 (1934)], let A denote an arbitrary family of subsets of M, and associate with A the family A^* consisting of all subsets of M whose intersection with every member of A is at most finite. Always, $A \subset (A^*)^*$. If $A^* = (A^*)^*$, A is called complete. Each complete family A of subsets of M gives rise to a group $G(A, G_a, G_b, \cdots)$. The elements of the group G are all possible systems of elements $(g_a, g_b, \cdots)$, where $g_a \in G_a$ and g_a is different from the identity of G_a only for subscripts belonging to a set of the family A. The composition of two elements of G is defined by the composition of the coordinates of these elements. This group is topologized as follows. Let L be a set of A^*. In each group G_a whose index is in L, select a neighborhood of the identity arbitrarily and denote it by U_a. Now define as a neighborhood of the identity in G the set of all elements whose coordinate $g_a \in U_a$ for indices a belonging to L, but which are not otherwise restricted (except that they belong to G).

Theorem 1. Let $G_a, G_b, \cdots$ be a countable system of Abelian topological groups, $X_a, X_b, \cdots$ the associated character groups, and A a complete set (of sets of indices). Then the group $A^*(X_a, X_b, \cdots)$ is the group of characters of $A(G_a, G_b, \cdots)$. Theorem 2. If the topological groups $G_a, G_b, \cdots$ are topologically complete groups then the group $A(G_a, G_b, \cdots)$ is complete if the system A is complete. Two important special cases of the general construction are the case of the topological product of the entire family of groups $G_a, G_b, \cdots$, and the case of their algebraic direct product. For the first of these the set A is the set of all subsets of M, and for the second it is the set of all finite subsets of M. The author indicates a construction of a complete Borel hierarchy of groups, starting with some countable family. In addition to some theorems on this and other matters, the author formulates the following two lemmas. (1) Let G be a compact Abelian group satisfying the second countability axiom and H an everywhere dense subgroup. Then there is a compact subset F of H such that if a character χ satisfies $|\chi(f)| < 1/20$ for every $f \in F$, then $\chi = 0$. (2) If H is an everywhere dense subgroup of the compact Abelian group G (with second countability) and X is the character group of H, then X coincides with the character group of G.

L. Zippin (Flushing, N. Y.).

***Bohr, Harald, and Følner, Erling. On a structure theorem for closed modules in an infinite-dimensional space.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 45–62. Interscience Publishers, Inc., New York, 1948. $5.50.

The group (under componentwise addition) of all sequences of real numbers $x_1, x_2, \cdots$ may be metrized in such a manner that the sequence of sequences $x_1^n, x_2^n, \cdots$ converges to the sequence $x_1, x_2, \cdots$ if and only if x_j^n converges to x_j for all $j = 1, 2, \cdots$. The authors call the resulting topological group R^∞. If I and J are disjoint sets of positive integers, let $S(I, J)$ denote the set of all sequences $x_1, x_2, \cdots$ in R^∞ such that $x_j = 0$ for each $j \in I$ and x_j is an integer for each $j \in J$. The principal result of this paper is that a subset M of R^∞ is a closed subgroup if and only if it is the image of some $S(I, J)$ by some bicontinuous automorphism of R^∞. It is proved via a similar theorem for a group "dual" to R. Since the appearance of this paper the sort of duality involved here has been discussed more generally by S. Kaplan [Duke Math. J. **15**, 649–658 (1948); these Rev. **10**, 233]. For closely related duality considerations see also Katětov [Acta Fac. Nat. Univ. Carol., Prague no. **181** (1948); these Rev. **10**, 127]. *G. W. Mackey* (Chicago, Ill.).

Gleason, A. M. Square roots in locally Euclidean groups. Bull. Amer. Math. Soc. **55**, 446–449 (1949).

Let G be a locally Euclidean topological group of dimension n. If N is a given neighborhood of the identity, then there is a neighborhood M of the identity such that every element in M has a square root in N. This result was also proved by Kerékjártó [C. R. Acad. Sci. Paris **193**, 1384–1385 (1931)]. If G does not have arbitrarily small subgroups then there are neighborhoods M and N of the identity such that every element in M has a unique square root in N. It is also shown that the inverse mapping has degree $(-1)^n$.

D. Montgomery (Princeton, N. J.).

Mal'cev, A. I. On a class of homogeneous spaces. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 9–32 (1949). (Russian)

This paper is devoted to the study of the coset spaces of nilpotent Lie groups. First there is a reduction to the compact case: if a connected nilpotent Lie group G acts transitively on a space M, then M is the topological product of a Euclidean space and a compact space acted on transitively by a connected subgroup of G. If a connected nilpotent Lie group G acts transitively and effectively on a compact space M, then M is a coset space of G modulo a discrete subgroup. A compact coset space of the form G/H, where G is a connected, simply connected nilpotent Lie group and H is discrete, is called a nilmanifold; H is the fundamental group of the manifold. The rest of the paper is devoted to nilmanifolds. Two nilmanifolds with isomorphic fundamental groups are homeomorphic. In order that a given abstract group be the fundamental group of a suitable nilmanifold, it is necessary and sufficient that it be a finitely generated nilpotent group with no elements of finite order. In order for a connected, simply connected nilpotent Lie group G to give rise to a nilmanifold, it is necessary and sufficient that the Lie algebra of G admit a basis with respect to which the structure constants are rational. In order that two nilmanifolds correspond in this fashion to isomorphic rational Lie algebras, it is necessary and sufficient that they be finite-sheeted coverings of a third nilmanifold. *I. Kaplansky.*

Mal'cev, A. I. Nilpotent torsion-free groups. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 201–212 (1949). (Russian)

The groups under discussion have no elements of finite order, and have a finite central series. In such a group extraction of nth roots is unique when possible; if it is always possible, the group is said to be complete. The members and factor-groups of the ascending central series of G are nilpotent and torsion-free along with G; this is true for the descending central series if G is complete, but not in general otherwise. If not complete, G may be embedded in a unique completion G^*, the rank of which is at most equal to the rank of G (it may be smaller). The connection between subgroups of G and G^* is studied. There is a one-to-one correspondence between complete nilpotent torsion-free groups of finite rank and Lie algebras of finite

order over the rational numbers, and the usual kind of correspondence is valid between subalgebras and complete subgroups. Finally it is indicated that some of the results may be extended to locally nilpotent groups.

Extensive use is made of an earlier paper of the author [see the preceding review]; thus topology intervenes in the proofs of the above theorems, although their statement is purely algebraic. In the paper cited, the author has shown that any finitely generated nilpotent torsion-free group can be exhibited as the fundamental group of a homogeneous space of a connected nilpotent Lie group, and this makes it possible to apply Lie group theory. Besides this, the principal tool in the proofs is the commutator calculus.

I. Kaplansky (Chicago, Ill.).

Tamari, Dov. Caractérisation des semi-groupes à un paramètre. C. R. Acad. Sci. Paris **228**, 1092–1094 (1949).

Let $\mathcal{G}$ be a one-parameter semi-group, that is: (1) $\mathcal{G}$ is closed under an associative operation xoy with cancellation law on both sides; (2) $\mathcal{G}$ is a connected one-dimensional locally Euclidean space, hence homeomorphic with either an interval I of the real axis or the circle C; (3) xoy is continuous in x and y together. By showing that any such $\mathcal{G}$ can be embedded in a group, and using É. Cartan's theorem on one-parameter groups [La théorie des groupes finis et continus et l'analysis situs, Mémor. Sci. Math., no. 42, Gauthier-Villars, Paris, 1930, p. 11], the author shows that $\mathcal{G}$ must be isomorphic with exactly one of the following: (1) additive group R^1 of reals; (2) reals$\geqq 0$ (or >0); (3) reals$\leqq 0$ (or <0); (4) reals$\geqq a_1>0$ (or $>a_1>0$); (5) reals $\leqq a_2<0$ (or $<a_2<0$); (6) additive group T^1 of reals mod 1.

A. H. Clifford (Baltimore, Md.).

Tamari, Dov. Groupoïdes reliés et demi-groupes ordonnés. C. R. Acad. Sci. Paris **228**, 1184–1186 (1949).

After a very general discussion of "groupoïdes reliés (ordonnés) à dr.," the author turns to the theory of a "demi-groupe D simplement ordonné à dr." This means: (1) D is closed under an associative operation xy; (2) D is a simply ordered set; (3) $D=D_0\cup D_1\cup D_{-1}$, where D_0 consists of the "right divisors of zero" ($a\varepsilon D_0$ if $ba=ca$ for some $b\neq c$ in D), D_1 consists of the "conservateurs à dr." ($a\varepsilon D_1$ if $x>y$ implies $xa>ya$), and D_{-1} consists of the "inverseurs à dr." ($a\varepsilon D_{-1}$ if $x>y$ implies $xa<ya$). If $D=D_1$, D is called conservative. Some of the results announced are the following. If C is a conservative demi-group, and $M_2=\{1,\epsilon\}$ the group of order 2, the direct product $D=C\times M_2$ can be simply ordered so that $D_1=C$ and $D_{-1}=C\epsilon$. If D is a group, either $D_1=D$ or D_1 is a subgroup of index two. Let D be a demi-group which is regular on the right in the sense of Ore [Ann. of Math. (2) **32**, 463–477 (1931)], and let it be embedded, by Ore's construction, in a "right quasi-group" $Q(D)$, i.e., a demi-group in which every right non-divisor of zero has a right inverse. Then $Q(D)$ can be simply ordered on the right in exactly one way preserving the given ordering in D.

A. H. Clifford (Baltimore, Md.).

Croisot, Robert. Hypergroupes partiels. C. R. Acad. Sci. Paris **228**, 1090–1092 (1949).

Un hypergroupe partiel est un système à opération associative, non uniforme et non toujours définie, tel que tout élément ait un inverse à droite unique et un inverse à gauche unique. Un hypergroupe partiel est dit inversable si $\alpha\beta\gamma$ entraine $\alpha''\gamma\beta$ et $\gamma\beta'\alpha$ (α'' inverse à gauche de α et β' inverse à droite de β). Il en résulte que l'inverse est bilatère et que $\alpha\beta\gamma$ équivaut à $\bar{\beta}\bar{\alpha}\bar{\gamma}$ ($\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ inverses de α, β, γ). L'auteur étudie ensuite les hypergroupes satisfaisant à des conditions telles que les suivantes. Si ϵ et ϵ' sont deux unités, il existe au moins un scalaire tel que $\epsilon\sigma$ et $\sigma\epsilon'$ existent. Un tel hypergroupe partiel est le produit d'un groupoide de Brandt par un hypergroupe inversable.

J. Kuntzmann.

Croisot, Robert. Algèbres de relations et hypergroupes partiels. C. R. Acad. Sci. Paris **228**, 1181–1182 (1949).

L'ensemble E de tous les sous ensembles d'un hypergroupe partiel inversable M est une algèbre de relations généralisée A (c'est-à-dire une algèbre de Boole complète sur laquelle est définie une multiplication) atomique si l'on prend pour relation d'ordre l'inclusion dans E et pour multiplication celle induite par M. Réciproquement, toute algèbre de relations généralisée atomique admet une telle représentation isomorphe. Une algèbre de relations A_1 est une algèbre définie sur un multigroupe M_1 tel que: si ϵ et ϵ' sont des unités, il existe au moins un α pour lequel $\epsilon\alpha$ et $\alpha\epsilon'$ sont définis [voir l'analyse ci-dessus]. Toute algèbre A atomique est isomorphe d'une manière unique à un produit d'algèbres A_1 atomiques. L'auteur caractérise ensuite parmi les algèbres A_1 l'algèbre de toutes les relations sur ensemble.

J. Kuntzmann (Grenoble).

Siverceva, N. On the simplicity of the associative system of singular square matrices. Mat. Sbornik N.S. **24(66)**, 101–106 (1949). (Russian)

The author follows Liapin [Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 497–515 (1947); these Rev. **9**, 134] in the use of the term "associative system," except that she does not assume a unit element. The definition of normality is accordingly modified: a subsystem N of S is normal if for $a,b\varepsilon S$ and $n\varepsilon N$ the elements of the pairs (anb, ab), (an, a), and (nb, b) both belong or both do not belong to N. It is shown that the system of all singular square matrices is simple in the sense of having no nontrivial normal subsystems. More generally, if S is a system of matrices containing all singular matrices, then any normal subsystem which contains a singular matrix is all of S. The proof is computational in character.

I. Kaplansky.

NUMBER THEORY

*Pizá, Pedro A. **Arithmetical Essays.** Imprenta Soltero, Santurce, Puerto Rico, 1948. vi+173 pp. (1 plate). $3.00.

This book contains four essays on unrelated subjects written by one who is "a self-taught enthusiastic amateur of arithmetic addicted to playing with numbers for fun." There are very few proofs, most of the results being obtained by induction from numerical examples, of which a great many are given. About half of the book is taken up by the first essay, entitled "Pascalian arithmetical tables." Here the author constructs generalizations of the well-known table of binomial coefficients by a process which is essentially equivalent to the inverse of taking differences. Thus he obtains formulas for such sums as $S_n(x)=1^n+2^n+\cdots+x^n$ in terms of binomial coefficients, most of which would be readily derivable from Newton's formula. A somewhat different type of formula is illustrated by

$$S_n(x)=\sum_{c=1}^{n} B_c^{(n)}\binom{x+n+1-c}{n+1},$$

where the coefficients $B_c^{(n)}$ are determined by

$$B_c^{(n)}=cB_c^{(n-1)}+(n-c+1)B_{c-1}^{(n-1)},\quad B_1^{(1)}=1,\quad B_c^{(1)}=0\quad (c>1).$$

It is claimed that these formulas are more adapted to numerical computation than are the classical formulas involving the Bernoulli numbers. In the second essay the so-called Kummer coefficients are discussed. These are numbers of the form $n(n-c)(n-c-1)\cdots(n-2c+3)/(c-1)!$ and are the coefficients that arise when x^n+y^n is expressed as a polynomial in x, y, and $z=x+y$, according to a formula due to Kummer. Many relations, most of them trivial, are given between these numbers; perhaps the most noteworthy is the fact that the sum of all of them for a fixed n is the nth number in the Fibonacci series $1, 2, 3, 5, 8, \cdots$. The third essay deals with "escalator numbers"; these are numbers $a_1, \cdots, a_m$ having the property that $a_1a_2\cdots a_n=a_1+a_2+\cdots+a_n$ for $n=1, 2, \cdots, m$. It is easily seen that such a set of numbers can be constructed starting with any number $a_1\neq 1$; simple properties of these numbers are easily derived. In the last essay the author discusses "Glenie triangles," mainly with respect to their construction by ruler and compass. A Glenie triangle is one whose sides a, b, c satisfy the Fermat equation $a^n+b^n=c^n$ [see the author's Fermagoric Triangles, Polytechnic Institute of Puerto Rico, San Germán, P. R., 1945; these Rev. **8**, 313] and which is to be inscribed in a given circle. It is clear that the constructions given in the book just quoted can be at once applied to construct these Glenie triangles and to find their properties.

H. W. Brinkmann (Swarthmore, Pa.).

Piza, Pedro A. Summation of powers of integers. Euclides, Madrid **8**, 255–260 (1948). (Spanish)

A brief description, with illustrations, of the methods used to compute such sums as $S_n(x)=1^n+2^n+\cdots+x^n$. These are explained in detail in the first essay of the book reviewed above. *H. W. Brinkmann* (Swarthmore, Pa.).

Mac Fhraing, Rob Alasdair. The numbering of Fionn's and Dubhan's men, and the story of Josephus and the forty Jews. Proc. Roy. Irish Acad. Sect. A. **52**, 87–93 (1948). (Gaelic. English summary)

The first part of the paper gives a Scottish Gaelic version, together with a mnemonic in verse, of the well-known story of the fifteen Christians and fifteen Turks [see Ball, Mathematical Recreations and Essays, 11th ed., Macmillan, London, 1939; New York, 1947, pp. 32–35; these Rev. **8**, 440]. The Ossianic hero Fionn (Fingal) and his men take the place of the Christians, and Dubhan and his men correspond to the unfortunate Turks. The general problem is as follows. The numbers $1, 2, \cdots, n$ are arranged in a circle and every mth number is deleted. It is required to find the last number to be deleted, $d_m(n)$. It was shown by P. G. Tait [Scientific Papers, v. 2, pp. 432–435 = Proc. Roy. Soc. Edinburgh **22**, 165–168 (1899)] that $d_m(n+1)\equiv m+d_m(n) \pmod{(n+1)}$, and the author uses this to formulate an algorithm by means of which $d_m(n)$ may be quickly determined. When $m=2$ this rule becomes especially simple; in fact $d_2(n)=2n+1-2^{e+1}$ where $2^e\leqq n<2^{e+1}$.

R. A. Rankin (Cambridge, England).

Armellini, G. Sui numeri perfetti. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 109–113 (1948).

[For part I cf. same Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 9–14 (1948); these Rev. **10**, 181.] It is proved that any number of the form $q^k\alpha^2\beta^2\cdots\gamma^2$, where $q, \alpha, \beta, \cdots, \gamma$ are different primes, cannot be perfect. This has already been proved by Steuerwald [S.-B. Bayer. Akad. Wiss. 1937, 68–72] and several other results of a similar type are known [see, for example, A. Brauer, Bull. Amer. Math. Soc. **49**, 712–718, 937 (1943); these Rev. **5**, 90]. *R. A. Rankin.*

Boos, Pierre. Divisibilité des polynomes relativement aux puissances d'un nombre entier. Bull. Soc. Math. France **76**, 65–78 (1948).

Let $g(x)=px^n+ax^{n-1}+a_2x^{n-2}+\cdots$ with p prime and $(a, p)=1$. It is proved that for every nonnegative integer k there exist polynomials $u_k(x)$ and $v_k(x)$ of degrees k and $n-1$ such that $gu_k\equiv v_k \pmod{p^{k+1}}$ and the coefficient of x^{n-1} in v_k is prime to p. For fixed $g(x)$ and k, the polynomial u_k is not unique: a different polynomial u_k' has the same property if and only if $u_k'\equiv\theta u_k \pmod{p^{k+1}}$, where θ is an integer prime to p. Relations are shown between the coefficients of u_k and u_j with $k\neq j$. Also there exists a linear polynomial $w_k(x)$ such that $g\equiv v_kw_k \pmod{p^{k+1}}$. A method is given for constructing this factoring of $g(x)$. All results are extended to the case of a polynomial g with leading coefficient p as before, further coefficients divisible by p, the coefficient of x^{n-k} being the first which is prime to p. *I. Niven.*

Bunickij, Eugen. Étude des fractions continues suivant un module entier $m>1$. Časopis Pěst. Mat. Fys. **73**, 109–119 (1949). (French. Czech summary)

Let $[a_1, \cdots, a_n]$ be a continued fraction and $P_1/Q_1, \cdots, P_n/Q_n$ the successive convergents with $(P_i, Q_i)=1$. In a complete residue system modulo m, let α_i, p_i, q_i be the residues of a_i, P_i, Q_i, respectively. The natural correspondence between $(\alpha_1, \cdots, \alpha_n)$ and the array

$$\begin{pmatrix} p_1 & \cdots & p_n \\ q_1 & \cdots & q_n \end{pmatrix}$$

is one-to-one. From the congruence $q_{r-1}p_r-p_{r-1}q_r\equiv(-1)^r \pmod m$, tables and recursion formulas are obtained which lead to the evaluation of the number of arrays having a given last column. This is carried out explicitly for $m=3$.

L. Tornheim (Ann Arbor, Mich.).

Pipping, Nils. Diagonalkettenbrüche. Acta Acad. Aboensis **16**, no. 5, 23 pp. (1949).

This paper is a sequel to a previous one [same Acta **15**, no. 10 (1947); these Rev. **8**, 501] in which certain properties of the elements of the semi-regular continued fraction known as the diagonal continued fraction were observed for the expansion of $D^{\frac{1}{2}}$ (D a nonsquare integer). In the present paper two of these properties are proved. These relate to the symmetry of the incomplete numerators and denominators in the period. The expansions of $(a^2-d)^{\frac{1}{2}}$ and $(a^2+d)^{\frac{1}{2}}$ for small d often differ only in the signs of the incomplete numerators. This property is shown to hold in certain special cases. The paper concludes with a table of the diagonal continued fraction expansion of D for $500<D\leqq 1000$. This extends the table for $D\leqq 500$ given in the previous paper.

D. H. Lehmer (Berkeley, Calif.).

Furquim de Almeida, Fernando. On a formula of Cipolla. Summa Brasil. Math. **1** (1946), no. 10, 207–219 (1948). (Portuguese)

M. Cipolla [Math. Ann. **63**, 54–61 (1907)] gave an explicit formula for the solutions of $x^n\equiv a \pmod p$, where p is an odd prime, $p-1=nh$ and $a^h\equiv 1 \pmod p$, as a polynomial $x=\sum A_ka^k$. Here the A_k were explicitly given in terms of a set $r_1, \cdots, r_h$ of integers whose nth powers are incongruent

modulo p, in particular in terms of the set $\gamma^r\delta^s$, where γ and δ appertain to the exponents m and $(p-1)/m$, m being a multiple of n such that m and $(p-1)/m$ are coprime ($r=0, 1, \cdots, m/n-1$; $s=0, 1, \cdots, (p-1)/m-1$). A modified exposition of these results is given here, and the following is derived. An explicit solution of the congruence $x^n \equiv a \pmod p$ is $x=a^\mu\gamma^{-\nu}$, where μ and ν are integers such that $n\mu-1=\nu(p-1)/m$, and r is a solution of the congruence $\gamma^{nr}\equiv a^{(p-1)/m} \pmod p$, $0\leqq r<m/n$. An indication is given of how the latter congruence can be solved.

G. Pall (Chicago, Ill.).

Moessner, Alfred. Verschiedene zahlentheoretische Untersuchungen und Diophantische Probleme. Bull. Calcutta Math. Soc. **40**, 147–152 (1948).

Numerical solutions of some special systems of Diophantine equations. *N. G. W. H. Beeger* (Amsterdam).

Wright, E. M. The Prouhet-Lehmer problem. J. London Math. Soc. **23**, 279–285 (1948).

The problem to which the title refers is that of finding sets of integers having equal sums of like powers. More precisely one seeks to solve the system of $k(s-1)$ simultaneous Diophantine equations

$$(1)\qquad \sum_{i=1}^{j}(x_{i1})^h=\sum_{i=1}^{j}(x_{i2})^h=\cdots=\sum_{i=1}^{j}(x_{is})^h \qquad (h=0, 1, \cdots, k)$$

for the s sets of j integers x_{iu}. To avoid triviality one requires that no set of x's is a mere permutation of another set. The case of $s=2$ is known as the Tarry-Escott problem [1912]. The general case was considered by E. Prouhet as long ago as 1851 [C. R. Acad. Sci. Paris **33**, 225]. Three numbers $P(k, s)$, $L(k, s)$, $W(k, s)$ are defined as follows: P is the least value of j for which (1) has a solution; L is the least value of j for which (1) has a solution such that not all the s sums (2) $\sum_{i=1}^{j}(x_{iu})^{k+1}$ ($u=1, 2, \cdots, s$) are equal, that is, the sets fail to have equal sums of $(k+1)$th powers. Finally W is the least value of j for which (1) has a solution such that the s sums (2) are distinct. Obviously $P\leqq L\leqq W$. Prouhet showed that $P(k, s)\leqq s^k$. Recently the author has shown that $P(k, s)\leqq(k^2+k+2)/2$ [Bull. Amer. Math. Soc. **54**, 755–757 (1948); these Rev. **10**, 101]. The present paper obtains similar results for L and W. Lehmer proved that $L(k, s)\leqq s^k$ [Scripta Math. **13**, 37–41 (1947); these Rev. **9**, 78]. The author proves that

$$L(k, s)\leqq(k+1)H(k, 2),$$
$$W(k, s)\leqq(k+1)H(k, s),$$

where $H(k, s)$ is the greatest integer not exceeding

$$\log((k+1)(s+1)/2-k+k^{-1})/\log(1+k^{-1}),$$

which is asymptotic to $k\log k$ for large k. Whether $L(k, s)$ is ever equal to $W(k, s)$ is an open question.

D. H. Lehmer (Berkeley, Calif.).

Swift, J. D. Diophantine equations connected with the cubic Fermat equation. Amer. Math. Monthly **56**, 254–256 (1949).

All solutions of $u^3-z^3=3uv$ are given in terms of 3 parameters, 3 sets of formulas being employed. Further results are then obtained by using known theory of the Fermat equation $x^3+y^3=z^3$, derived from the previous equation by the substitution $u=x+y$, $v=xy$. An application is also made to the cubic Fermat equation in quadratic fields.

I. Niven (Eugene, Ore.).

Gloden, A. A solution of a Diophantine equation. Scripta Math. **14**, 185–186 (1948).

It is announced without proof that formulas [to be published elsewhere] have been derived for solving $\sum_{i=1}^{3}(x_i^4-y_i^4)=0$ and the equation obtained by the substitution $x_i-y_i=2u_i$, $x_i+y_i=2v_i$. Examples involving two free parameters are given. *I. Niven* (Eugene, Ore.).

Campbell, R. C. A simple solution of the Diophantine equation $x^3+y^3=z^2+t^2$. Bull. Amer. Math. Soc. **55**, 442–446 (1949).

By completely factoring the indicated equation in the field $R(i, \sqrt{3})$ the author obtains a complete solution in rational numbers in terms of parameters λ, p, q, r, thus: $x=\lambda^2uvw(p+r)$, $y=2\lambda^2ruvw$, $z=\lambda^3ru^2v^2w$, $t=\lambda^3qu^3v^3w$, where $u=p+3r$, $v=p^2+3r^2$, $w=r^2+q^2$. Since x, y, z, w are invariant under the transformation $p=kp'$, $q=kq'$, $r=kr'$, $\lambda=k^{-2}\lambda'$, clearly p, q, r can be taken as integral parameters.

I. Niven (Eugene, Ore.).

Karanikolov, Chr. On a class of indeterminate equations. Spisanie Bulgar. Akad. Nauk. **65**, 291–293 (1942). (Bulgarian)

Cf. Tchakaloff and the author, C. R. Acad. Sci. Paris **210**, 281–283 (1940); these Rev. **1**, 200.

Châtelet, François. Relations entre l'arithmétique et la géométrie sur une quadrique. Bull. Soc. Math. France **76**, 108–113 (1948).

The author discusses for rational quadrics in S_3 the classical arithmetical results on the existence of rational points and the division of such quadrics into classes whose elements are transformable into one another by rational homographies of the space. His innovation is to use the representation of the reguli of a quadric by conics in S_3 to reduce the problems to known questions concerning conics over the rational field and quadratic extensions of it.

D. B. Scott (London).

Segre, B. Proprietà algebriche ed aritmetiche di forme cubiche. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 346–352 (1948).

In this and two other papers [see the following reviews], the author considers rational solutions of $x_1^3+\cdots+x_{r+2}^3=0$. This equation defines a cubic hypersurface F_r in S_{r+1}, which is invariant under a group G of $3^{r+1}(r+2)!$ collineations. In this paper the case $r+2=2n$ is considered. The hypersurface F is shown to contain $3^n(2n)!/2^n n!$ S_{n-1}'s, obtainable from $x_1+x_{n+1}=\cdots=x_n+x_{2n}=0$ by transforming by elements of G. Of these S_{n-1}'s, $(2n)!/2^n n!$ are rational. By projecting F onto an S_{2n-2} from two suitably chosen skew S_{n-1}'s of F, F can be parametrized rationally in quadratic forms in $2n-1$ variables. This parametrization gives all rational points of F. *R. J. Walker* (Ithaca, N. Y.).

Segre, B. Proprietà algebriche ed aritmetiche di forme cubiche. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 509–515 (1948).

On the cubic hypersurface F, $x_0^3+x_1^3+\cdots+x_{2n}^3=0$, there are $3^{n-2}(2n+1)!/2^{n-2}(n-2)!5!$ doubly infinite systems of S_{n-1}'s. Each system generates a V_{n-1}^3 which is a section of F by an S_{n-2}. The S_{n-2} is given by (*) $x_3+x_{n+3}=\cdots=x_n+x_{2n}=0$ or a transform of this by an element of G. [See the preceding review.] The corresponding section of F is the cubic defined by (*) and (**) $x_0^3+x_1^3+x_2^3+x_{n+1}^3+x_{n+2}^3=0$. The

determination of the S_{n-1}'s on F is thus reduced to the determination of the doubly infinite set of lines on the F_3 given by (**). A birational representation of these lines as an 18-fold plane is given. *R. J. Walker* (Ithaca, N. Y.).

Segre, B. Proprietà algebriche ed aritmetiche di forme cubiche. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 629–632 (1948).

A simple geometric proof is given of the following theorem due to S. Wachs [Revue Sci. (Rev. Rose Illus.) **80**, 402–406 (1942); these Rev. **7**, 169]. The only rational lines on the F_3 given by (**) [see the preceding review] are the 15 lines $x_i=x_j+x_k=x_l+x_m$, i, j, k, l, m being a permutation of 0, 1, 2, $n+1$, $n+2$. It follows that the only rational S_{n-1}'s on F_r, $r=2n-1$, are the $(2n+1)!/2^n n!$ spaces $x_{i_0}=x_{i_1}+x_{i_{n+1}}=\cdots=x_{i_n}+x_{i_{2n}}=0$, $i_0, \cdots, i_{2n}$ being a permutation of $0, \cdots, 2n$. For the case $n\geqq 5$ use is made of the existence of nontrivial integer solutions of $a_1x_1^2+\cdots+a_6x_6^2=0$, the a_i being integers not all of the same sign, to show that every rational point on F_r is either a rational point of a rational S_{n-1} or the residual intersection of a rational line tangent to F_r at such a point. From this result is obtained a complete rational parametrization of F_r in 10-ic forms in $2n+3$ variables. *R. J. Walker* (Ithaca, N. Y.).

Segre, B. Proprietà algebriche ed aritmetiche di forme cubiche. IV. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 3–11 (1948).

[Cf. the preceding review.] Let Φ_m be the cubic hypersurface in S_{m+1} defined by

$$x_1^3+\cdots+x_{m+3}^3=0, \quad x_1+\cdots+x_{m+3}+\lambda x_{m+3}=0,$$

where $\lambda=1$ if m is odd and $\lambda=4^{\frac{1}{3}}$ if m is even. It is shown that Φ_m has $(_{1(m+1)/m}^{m+2})$ double points. For $m=1$, 2, 3, or 4, Φ_m is a well-known variety, which has the maximum number of double points for its order and dimension. The situation of the double points of Φ_m is investigated, and a complete rational parametrization of Φ_m is obtained.

R. J. Walker (Ithaca, N. Y.).

Blaney, Hugh. Indefinite quadratic forms in n variables. J. London Math. Soc. **23**, 153–160 (1948).

Let $Q=Q(x_1, \cdots, x_n)$ denote an indefinite quadratic form in n variables with determinant $D\neq 0$. (I) For any $c\geqq 0$ a number $C=C(c, n)$ exists such that, for any Q, integers $x_1, \cdots, x_n$ with $(x_1, \cdots, x_n)=1$ exist, such that $c|D|^{1/n}<Q\leqq C|D|^{1/n}$. (II) For any $\gamma\geqq 0$ a number $\Gamma=\Gamma(\gamma, n)$ exists such that, for any Q and any real numbers $\alpha_1, \cdots, \alpha_n$, integers $x_1, \cdots, x_n$ exist, such that

$$\gamma|D|^{1/n}<Q(x_1+\alpha_1, \cdots, x_n+\alpha_n)\leqq\Gamma|D|^{1/n}.$$

Some applications to asymmetric approximations are given.

J. F. Koksma (Amsterdam).

Davenport, H. Note on indefinite ternary quadratic forms. J. London Math. Soc. **23**, 199–202 (1948).

The author proves the following theorem. Let $f(x, y, z)$ be an indefinite ternary form whose determinant D is negative. Denote the lower bound of the positive values of f for integral x, y, z by $L|D|^{\frac{1}{3}}$. Then for any real x_0, y_0, z_0 the lower bound of $|f(x_0+x, y_0+y, z_0+z)|$ for integral x, y, z does not exceed $\{2L^{\frac{1}{3}}-\frac{1}{4}L^2\}^{\frac{1}{3}}|D|^{\frac{1}{3}}$. From this theorem the author further deduces that there exist such forms f for which $f(x_0+x, y_0+y, z_0+z)$ takes arbitrarily small positive values, which is a contrast to a forthcoming result of the author on binary quadratic forms. *J. F. Koksma.*

Jarník, Vojtěch. On Estermann's proof of a theorem of Minkowski. Časopis Pěst. Mat. Fys. **73**, 131–140 (1949). (English. Czech summary)

Let M be a convex closed bounded n-dimensional set of points, symmetrical with respect to the origin, and having interior points. Let V denote the volume of M, and τ_i the least positive number τ for which the set τM contains at least i linearly independent lattice points (i.e., points with integer coordinates). A theorem of Minkowski states that (1) $\tau_1\tau_2\cdots\tau_nV\leqq 2^n$. Another theorem of Minkowski [Geometrie der Zahlen, part 1, Teubner, Leipzig, 1896, pp. 235–236] contains necessary and sufficient conditions for the equality sign to hold in (1). The author gives a new proof of the latter theorem. *T. Estermann* (London).

Chabauty, Claude. Géométrie des nombres d'ensembles non convexes. C. R. Acad. Sci. Paris **227**, 747–749 (1948).

Let S be a set of points in n-dimensional space, which is a star set relative to the origin, and let Δ denote the critical determinant of S. Let G be a lattice of determinant D. The author considers the successive minima $\mu_1, \cdots, \mu_n$ defined as follows: μ_k is the upper bound of all numbers $\mu\geqq 0$ for which the points of G in the set μS have a linear dimension less than k. His main theorem is that $\Delta\mu_1\cdots\mu_k\leqq D\psi(n)$. Here $\psi(n)$ is a function of n only, defined as follows. For any real $x_1, \cdots, x_n$ with $0<x_1\leqq x_2\leqq\cdots\leqq x_n$, take numbers $y_1, \cdots, y_n$ with $0<y_k<x_k$ for which y_{k+1}/y_k is an integer, such that $y_1\cdots y_n$ is as large as possible. Then $\psi(n)$ is the upper bound of $(x_1\cdots x_n)/(y_1\cdots y_n)$ for all sets x. The author notes that $\psi(2)=\sqrt{2}$, and that $\psi(n)<2^{n-C}/n^{\log 2}$, where C is Euler's constant. The method of proof of the main result is essentially that of Jarník and Knichal [Rozpravy II. Třídy České Akad. **53**, no. 43 (1943); these Rev. **8**, 565]. Errata: in the definition of $\gamma(S)$, "Sup" should stand first on the right; in the definition of $\alpha(S)$, the symbol A should be replaced by S, and $\alpha(S)$ should strictly be $\alpha(S, G)$; on page 749, $O(n)$ should be $o(1)$. [See the three following reviews.] *H. Davenport* (London).

Rogers, C. A. The product of the minima and the determinant of a set. Nederl. Akad. Wetensch., Proc. **52**, 256–263 = Indagationes Math. **11**, 71–78 (1949).

Let S be any set of points in n-dimensional space, and let $\Delta(S)$ be its critical determinant, i.e., the lower bound of the determinants of all lattices with no point different from O in S. For any lattice Λ, of determinant $d(\Lambda)$, define the successive minima $\mu_1, \cdots, \mu_n$ as follows: μ_k is the lower bound of the positive numbers μ such that the set μS contains at least k linearly independent points of Λ. [If S is a star set, this definition agrees with that of Chabauty in the preceding review.] The author's main theorem is that

$$(*) \qquad \mu_1\cdots\mu_n\Delta(S)\leqq 2^{\frac{1}{2}n(n-1)}d(\Lambda).$$

The author states that this refinement of a result of Jarník is implicit in an earlier paper [submitted to Proc. London Math. Soc. in 1946 but not yet published]. The inequality is strict when S is a bounded star body. [See the two following reviews.] *H. Davenport* (London).

Chabauty, Claude. Géométrie des nombres d'ensembles non convexes. C. R. Acad. Sci. Paris **228**, 796–797 (1949).

The author proves that the function $\psi(n)$, defined in his previous paper [see the second preceding review], has the value $2^{\frac{1}{2}n(n-1)}$, thus obtaining the inequality (*) above. The

only difference between this result and that of Rogers is that here the set is supposed to be a star set. The author also gives a construction for a star set S in n-dimensional space for which the inequality becomes an equality. Let C_k $(k=1, \cdots, n)$ be the set of all points $(tg_1, \cdots, tg_k, 0, \cdots, 0)$ for which $g_1, \cdots, g_k$ are integers with H.C.F. unity, $g_k \neq 0$, and $t \geqq 2^{(n-k)/n}$. The set S consists of all points which are not in any of $C_1, \cdots, C_n$, and the lattice is that of all points with integral coordinates. By modifying S, a bounded star body is constructed for which the ratio of the two sides of (*) is arbitrarily near to 1. The proofs are naturally somewhat condensed. [See also the following review.]

H. Davenport (London).

Mahler, K. On the minimum determinant of a special point set. Nederl. Akad. Wetensch., Proc. **52**, 633–642 (1949).

The author constructs a set S, as described in the preceding review, and so proves that the inequality (*) is best possible. There is also a modification of the construction, so as to obtain a bounded star body for which the ratio of the two sides of (*) is arbitrarily near to 1. The proof is detailed, and is accompanied by an investigation of the critical lattices of the set S. *H. Davenport* (London).

Rogers, C. A. The product of n homogeneous linear forms. J. London Math. Soc. **24**, 31–39 (1949).

Let $x_1, \cdots, x_n$ be n real homogeneous linear forms in n integral variables $u_1, \cdots, u_n$ with determinant unity. Let M be the lower bound of $|x_1 \cdots x_n|$ for all sets of integers $(u_1, \cdots, u_n)$ other than $(0, \cdots, 0)$. The author obtains an upper bound for M which is asymptotic to $(2\pi)^{\frac{1}{2}}(\frac{1}{2}+\log 2)n^{\frac{1}{2}}(e\sqrt{e})^{-n}$ for large n. This number is less than the corresponding bounds which have been obtained by Blichfeldt and the reviewer. However, in a note added before going to press the author observes that an upper bound for M, which is less than his, is contained implicitly in a paper by Blichfeldt [Monatsh. Math. Phys. **48**, 531–533 (1939); these Rev. **1**, 68]. The proof is based on a lemma which gives an upper bound for $\prod|z_r-z_s|/(\sum|z_i|)^{\frac{1}{2}m(m-1)}$ $(1 \leqq r<s \leqq m,\ 1 \leqq t \leqq m)$. The author has succeeded in sharpening this lemma, and he can now improve on Blichfeldt's result for large n. *R. A. Rankin.*

Chabauty, Claude. Sur le minimum du produit de formes linéaires réelles. C. R. Acad. Sci. Paris **228**, 1361–1363 (1949).

By using the method of a previous note [see the 5th preceding review] the author proves the following general theorem. Let S and T be two star sets in n-dimensional space, S being contained in T, with critical determinants $\Delta(S)$ and $\Delta(T)$. Let G be a critical lattice of T, and let λ be any number such that the points of G in the set λS have a linear dimension less than h. Then

$$\Delta(T) \geqq 2^{1-h}\lambda^{n+1-h}\Delta(S).$$

This provides a possible method of improving, in particular cases, on the obvious inequality $\Delta(T) \geqq \Delta(S)$. By taking S to be $|x_1|+\cdots+|x_n|<n$, and T to be $|x_1 \cdots x_n|<1$, and h to be 2, and by modifying a method of Davenport [see Nederl. Akad. Wetensch., Proc. **49**, 822–828 = Indagationes Math. **8**, 525–531 (1946); these Rev. **8**, 565], the author deduces the following theorem. Let $L_i(x_1, \cdots, x_n)$, where $i=1, \cdots, n$, be real linear forms of determinant 1. Then, provided n is sufficiently large, there exist integers $x_1, \cdots, x_n$, not all zero, such that

$$\left|\prod_{i=1}^{n} L_i(x_1, \cdots, x_n)\right|^{1/n} < \frac{1}{4.1}.$$

It may be mentioned, however, that a more precise result has been obtained by Rogers [see the preceding review]. [Errata, all on page 1362: (1) On line 10, omit n^{-1} in both formulae. (2) On line 8 from below, for $1/4n$ read $\frac{1}{4}$. (3) In the theorem, the possibility that $x_1=\cdots=x_n=0$ must be excluded, the statement "0<" must be deleted from the displayed formulae, and n must be deleted in each denominator.] *H. Davenport* (London).

Hinčin, A. Ya. On some applications of the method of the additional variable. Uspehi Matem. Nauk (N.S.) **3**, no. 6(28), 188–200 (1948). (Russian)

The method of the additional variable, introduced by L. J. Mordell [J. London Math. Soc. **12**, 34–36, 166–167 (1937)] is the following. Suppose that $L_1, \cdots, L_n$ are linear forms in any number of variables, and that it is required to prove that there exist integer values of the variables for which the L_i satisfy certain conditions; then one considers the set of $(n+1)$ forms $L_1-\alpha u, L_2, \cdots, L_n, u$, where u is the additional variable, and α is a suitably chosen real number. One proves in turn, (i) that the modified set of forms satisfy the required conditions for some integer u within known bounds, (ii) that the only value of u for which the conditions can in fact be satisfied is zero. Three applications of the method are made, as follows. A proof in strengthened form of a theorem of K. Mahler [Časopis Pěst. Mat. Fys. **68**, 85–92 (1939); these Rev. **1**, 202]. A simplified proof of a theorem of V. Jarník [Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **3**, 193–212 (1938)]. A greatly shortened proof, following a suggestion by Mahler, of a theorem of the reviewer [Proc. London Math. Soc. (2) **49**, 409–420 (1947); these Rev. **9**, 271]. *F. J. Dyson* (London).

Hinčin, A. Ya. On the fractional parts of a linear form. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 3–8 (1949). (Russian)

Let $\{z\}$ denote the distance of a real number z from the nearest integer. Let $S(x)=\theta_1x_1+\cdots+\theta_nx_n$ be a linear form with real coefficients in n integer variables x_i, components of an n-dimensional vector x. Let $N(x)=\max(|x_1|, \cdots, |x_n|)$. Then a classical theorem of Dirichlet [see J. F. Koksma, Diophantische Approximationen, Springer, Berlin, 1936, p. 5] states that for every $t \geqq 1$ there exists an integral vector x such that $\{S(x)\}<t^{-1}$, $0<N(x) \leqq t^{1/n}$. The author proves the following extension of Dirichlet's theorem. For every $t \geqq 1$ there exist n linearly independent integral vectors $x^1, \cdots, x^n$ such that

$$\{S(x^i)\}<t^{-1}, \qquad i=1, \cdots, n,$$

$$0<\prod_{i=1}^{n} N(x^i) \leqq c_n t,$$

where c_n is a constant depending only on n. The proof is elementary, making repeated use of the Dirichlet "Schubfachprinzip." *F. J. Dyson* (London).

Davis, C. S. Note on a conjecture by Minkowski. J. London Math. Soc. **23**, 172–175 (1948).

Minkowski [see Koksma, Diophantische Approximationen, Springer, Berlin, 1936, pp. 51–88] was led to the problem of finding the minimum possible area of a parallelogram with one vertex in the origin and the three remaining

The first reference should be read:
[Minkowski, Diophantische Approximationen, 2d ed.; Teubner, Leipzig, 1927, pp 51–58]

vertices on the curve $|\xi|^p+|\eta|^p=1$ ($p\geqq 1$). This problem is easy for $p=1$, $p=2$ and $p=\infty$. For $1<p<2$ Minkowski conjectured that the solution is given by a parallelogram with one vertex on a coordinate axis; its area is $\Delta_0=(1-2^{-p})^{1/p}$. For $p>2$ he conjectured that the minimum is given by a parallelogram with one vertex on one of the lines $\xi=\pm\eta$, this area being Δ_1 given by the equation $(2^{2/p}\Delta_1+1)^p+(2^{2/p}\Delta_1-1)^p=2^{p+1}$. Mordell proved the truth of this conjecture for $p=4$ [J. London Math. Soc. 16, 152–156 (1941); these Rev. 3, 167], and now the author proves that the conjecture is false in general. Minkowski remarked that Δ_0 and Δ_1 are clearly extrema as is seen from the symmetry of the curve; he assumed that there are no further extrema, and finally he made a statement which is equivalent to: $\Delta_0<\Delta_1$ for $1<p<2$; $\Delta_1<\Delta_0$ for $p>2$. The author proves that these inequalities are false and he conjectures: $\Delta_1<\Delta_0$ for $1<p<2$ and for $p>p_0$; $\Delta_0<\Delta_1$ for $2<p<p_0$, where p_0 is a certain value, about 2.5725.

J. F. Koksma (Amsterdam).

Macbeath, A. M. Non-convex regions in three and more dimensions. Proc. Cambridge Philos. Soc. **45**, 161–166 (1949).

Let R be the region of points $(x_1, \cdots, x_n)$ in Euclidean n-space for which $f(|x_1|, \cdots, |x_n|)\leqq k$, where $f(|x_1|, \cdots, |x_n|)$ is a function which is strictly increasing in each of the variables and has the property that $f(px+qy)\geqq k$, $p>0$, $q>0$, $p+q=1$, $f(x)=f(y)=k$. If $\lambda_1x_1+\cdots+\lambda_nx_n=1$ is a tac-hyperplane for a boundary point of R with positive coordinates, let $u=\varphi(c)$ be the nonnegative function defined for positive c by the relations $f(z_1, \cdots, z_{n-1}, c)\geqq k$, $u=\min(\lambda_1z_1+\lambda_2z_2+\cdots+\lambda_{n-1}z_{n-1})$. The author constructs a nonconvex subset R_2 of R whose volume V may be expressed as an integral involving the function $\varphi(c)$ and which has the property that, if a and b are vectors representing the points of R_2, the point defined by $a-b$ is in R. This follows readily from a variation of a theorem of Mordell [Nederl. Akad. Wetensch., Proc. **49**, 773–781, 782–792 = Indagationes Math. **8**, 476–484, 485–495 (1946); these Rev. **8**, 369] for plane domains. The principal result of the paper is to show that a lattice with determinant not exceeding V has a point, other than the origin, in R. This follows from the above results by applying a theorem of Blichfeldt [Trans. Amer. Math. Soc. **15**, 227–235 (1914)]. The result is applied to the region of points $(x_1, \cdots, x_n)$ for which $(|x_1|+c)(|x_2|+c)\cdots(|x_n|+c)\leqq(1+c)^n$, c nonnegative.

D. Derry (Vancouver, B. C.).

Erdös, P. Some remarks on Diophantine approximations. J. Indian Math. Soc. (N.S.) **12**, 67–74 (1948).

Let $d(m)$ be the number of positive divisors of m, let $r_k(m)$ be the number of representations of m as a sum of k squares, and let $\{\alpha\}$ be the distance from α to the nearest integer. Simple proofs are given here of the following relations, which hold for almost all real α:

$$\sum_{m=1}^{n} d(m)e^{2\pi im\alpha}=O(n^{\frac{1}{2}}\log n),$$

$$\sum_{m=1}^{n} r_3(m)e^{2\pi im\alpha}=O(n^{\frac{1}{2}}\log n),$$

$$\sum_{m=1}^{n} r_4(m)e^{2\pi im\alpha}=O(n\log^2 n).$$

More complicated proofs of these relations were given earlier by Chowla [Math. Z. **33**, 544–563 (1931)]. A conjecture of Spencer, that for almost all α

$$\sum_{m=1}^{n} 1/m\{m\alpha\}=(1+o(1))\log^2 n,$$

is also proved, and the following theorem (among others) is stated without proof. Let $f(n)$ be an increasing function of n for which $f(n)>(2+\epsilon)n\log n$ and $\sum_{n=1}^{\infty}1/f(n)$ converges. Then for almost all α and $n>n_0(\alpha)$, $\sum_{m=1}^{n}1/\{m\alpha\}<f(n)$. [There are many misprints: the first inequality in the third paragraph, p. 67, should be $|\alpha-a/b|<1/2b^2$; the right sides of (8) and (11) should be $O(n\log^{3+\epsilon}n)$ and $O(n\log^2 n)$ respectively; in footnotes 2, 3, p. 68, the years of publication are 1932 and 1931, respectively; the upper limit on the summation in the last displayed formula of the paper should be n, rather than ∞; p. 73, lines 7, 8, $(m\alpha)$ is to be replaced by $\{m\alpha\}$; in (31) there should be an equality sign before $o(\log^2 n)$.]

W. J. LeVeque (Ann Arbor, Mich.).

Obrechkoff, Nikola. Sur l'approximation des nombres irrationnels. C. R. Acad. Sci. Paris **228**, 352–353 (1949).

Si $0<\omega<1$ et si n est un nombre entier positif, il existe deux nombres entiers x et y tels que $|\omega x-y|<1/(n+1)$, $1\leqq x\leqq n$. Cette amélioration du théorème de Dirichlet se trouve déjà chez Minkowski [Koksma, Diophantische Approximationen, Springer, Berlin, 1936, pp. 5–6]. Si ω se trouve en dehors de quelques intervalles indiquées par l'auteur (dépendant de n) on peut remplacer dans l'inégalité mentionnée $1/(n+1)$ par $1/(n+2)$ (démontré à l'aide d'une suite spéciale de Farey). L'auteur donne un énoncé semblable pour l'inégalité

$$|\omega_1x_1+\cdots+\omega_mx_m-z|\leqq(n+1)^{-m},\qquad |x_i|\leqq n.$$

J. F. Koksma (Amsterdam).

Prasad, A. V. Note on a theorem of Hurwitz. J. London Math. Soc. **23**, 169–171 (1948).

Generalizing Hurwitz's well-known approximation theorem, the author proves the following theorem. Let p_n/q_n denote the nth convergent to $\xi_0=\frac{1}{2}(\sqrt{5}-1)$ starting with $p_1/q_1=1$. Let $C_m=\frac{1}{2}\{\sqrt{5}+1\}+p_{2m-1}/q_{2m-1}$ $(m=1, 2, \cdots)$. Then for any irrational number ξ the inequality

$$|\xi-p/q|\leqq C_m^{-1}q^{-2}$$

has at least m solutions in relatively prime integers p, q, with $q>0$. If $\xi=\xi_0$ there are not more than m such solutions.

J. F. Koksma (Amsterdam).

Brauer, Alfred, and Macon, Nathaniel. On the approximation of irrational numbers by the convergents of their continued fractions. Amer. J. Math. **71**, 349–361 (1949).

The authors prove numerous generalizations of the theorem of Borel [J. Math. Pures Appl. (5) **9**, 329–375 (1903)] that, if p_n/q_n is the nth convergent to the regular continued fraction expansion of the positive irrational number ξ, and if $|\xi-p_n/q_n|=1/\lambda_nq_n^2$, then for every n, $\max(\lambda_n, \lambda_{n+1}, \lambda_{n+2})>\sqrt{5}$. They show that at least $m+1$ of each set of $3m+2$ consecutive λ's exceed $\sqrt{5}$, or at least m exceed 3, and also that for every n,

$$\lambda_n\lambda_{n+1}>\lambda_n+\lambda_{n+1}>\max\{\lambda_n^2/(\lambda_n-1), \lambda_{n+1}^2/(\lambda_{n+1}-1)\}>4.$$

Lower bounds for sums of 3, 4 and 8 successive λ's are obtained, and it is deduced that

$$\liminf_{m\to\infty} m^{-1}\sum_{i=0}^{m-1}\lambda_i\geqq\frac{31+15\sqrt{5}}{32}>2.0169.$$

This is not shown to be the best possible constant; by Hurwitz's theorem [Math. Ann. 39, 279–284 (1891)] this lower limit cannot exceed $\sqrt{5}$ for some irrational numbers ξ. *W. J. LeVeque* (Ann Arbor, Mich.).

Todd, H. On Diophantine approximation to certain exponential and Bessel functions. Proc. London Math. Soc. (2) 50, 550–559 (1949).

It is shown that solutions of a certain type of linear difference equation of order $s+1$ provide simultaneous rational approximations to s irrationals, which are determined by the coefficients of the difference equation and the initial values chosen for the formation of the solutions. Applications are given for the case in which the irrationals thus defined are combinations of exponential or of Bessel functions. Theorems and examples are too complicated to be quoted here in detail. *J. F. Koksma* (Amsterdam).

Shapiro, Harold N. Note on a theorem of Dickson. Bull. Amer. Math. Soc. 55, 450–452 (1949).

For an integer n and any real number k, say that n is k-deficient if $kn>\sigma(n)$, the sum of the divisors of n, otherwise k-nondeficient. Say that n is primitive k-nondeficient if it is k-nondeficient but all proper divisors of n are k-deficient. It is proved that there is at most a finite number of primitive k-nondeficient numbers n having a fixed number of distinct prime factors provided either of the following conditions is satisfied: (1) k is irrational; (2) k is rational, say $k=r/s$, and if $p|r$ and $p^\alpha\|n$ then $\alpha<C$, a fixed constant. Condition (2) is satisfied if $(n, r)=1$. This generalizes L. E. Dickson's results for the case $k=2$. *I. Niven.*

Watson, G. N. A table of Ramanujan's function $\tau(n)$. Proc. London Math. Soc. (2) 51, 1–13 (1949).

This table gives the coefficient $\tau(n)$ of x^{n-1} in the 24th power of Euler's product $(1-x)(1-x^2)(1-x^3)\cdots$ for all integers $n\leqq 1000$. The most extensive previous table is that of Gupta for $n\leqq 400$ [Proc. Nat. Inst. Sci. India 13, 201–206 (1947); these Rev. 10, 104]. Besides $\tau(n)$, the function $\tau^*(n)=n^{-11/2}\tau(n)$ is given to 5 decimals. This function, according to the as yet unproved "Ramanujan hypothesis," should be less than 2 in absolute value for all prime values of n. The table shows that $\tau^*(p)$, for p a prime, lies between $\tau^*(103)=-1.91881$ and $\tau^*(479)=1.90410$, at least when $p<1000$. Let $\tau^*(p)=2\cos\pi r_p$; then, for $p<1000$, not only is r_p real, but also $\frac{1}{4}<r_p<\frac{3}{4}$ except for the 6 primes $p=103$, 313, 479, 619, 719, and 877. This table will be of much use in the further investigation of this interesting numerical function. *D. H. Lehmer* (Berkeley, Calif.).

v. d. Blij, F. S. Ramanujan's function $\tau(n)$. Math. Centrum Amsterdam. Rapport ZW 1948-010, 18 pp. (1948). (Dutch)

This report is a topical history of the function $\tau(n)$ of Ramanujan defined as the coefficient of x^{n-1} in the power series development of the 24th power of the product $(1-x)(1-x^2)(1-x^3)\cdots$. The report is in 7 sections whose titles might have been (1) introduction, congruence properties of $\tau(n)$; (2) properties of modular functions, especially Δ and η; (3) the Fourier coefficients of modular forms; (4) the associated Dirichlet series and their Euler products; (5) quadratic forms and their associated theta functions; (6) miscellaneous results on quadratic forms and their numbers of representations; (7) bibliography. The last section extends through 1947 and contains 73 titles by 33 authors. There are references to other problems related to those concerning $\tau(n)$. This section is an especially valuable feature of the report since no proofs (or even indications of proofs) are given in the text. The author presents the subject matter in clear and concise manner. The reviewer has noticed only one mistake. The erroneous identity

$$\sum_{n=1}^{\infty}\tau(n)e^{-ns^{\frac{1}{2}}}=(2\pi)^{25/2}\Gamma(25/2)\sum_{n=1}^{\infty}\tau(n)(s^2+4n\pi^2)^{-25/2}$$

is incorrectly quoted from Hardy ["Ramanujan," Cambridge University Press, 1940, p. 124; these Rev. 3, 71] who, in turn, has made an error. In fact the left member should read $2^{-\frac{1}{2}}s^{-1}\sum_{n=1}^{\infty}\tau(n)e^{-2ns^{\frac{1}{2}}}$. *D. H. Lehmer.*

Gupta, Hansraj. The vanishing of Ramanujan's function $\tau(n)$. Current Sci. 17, 180 (1948).

By using certain congruence properties of Ramanujan's function $\tau(n)$ the reviewer has shown that, for $n<3316799$, $\tau(n)\neq 0$ [Duke Math. J. 14, 429–433 (1947); these Rev. 9, 12]. By using recent results of Bambah and Chowla [Bull. Amer. Math. Soc. 53, 950–955 (1947); Proc. Nat. Inst. Sci. India 12, 433 (1946); J. London Math. Soc. 22, 140–147 (1947); these Rev. 9, 226, 331], the author asserts that $\tau(n)$ fails to vanish for $n<1791071999$. [An unpublished result of the reviewer is that this number may be replaced by the prime 214928639999.] *D. H. Lehmer* (Berkeley, Calif.).

Lahiri, D. B. Some non-Ramanujan congruence properties of the partition function. Proc. Nat. Inst. Sci. India 14, 337–338 (1948).

Let $p(n)$ denote the number of unrestricted partitions of n. The author has found three new congruences:

$$p(49m+k)=0 \pmod{49}, \qquad k=19, 33, 40.$$

These results follow from the identity of Ramanujan

$$\sum_{n=0}^{\infty}p(7n+5)x^n=7[f(x^7)]^3[f(x)]^{-4}+49x[f(x^7)]^7[f(x)]^{-8},$$

where $f(x)=(1-x)(1-x^2)(1-x^3)\cdots$, and the use of Jacobi's triangular number theorem concerning $[f(x)]^3$. Similar congruence properties do not hold for the moduli 25 and 121. *D. H. Lehmer* (Berkeley, Calif.).

Lahiri, D. B. Further non-Ramanujan congruence properties of the partition function. Science and Culture 14, 336–337 (1949).

Let $p(n)$ denote the number of unrestricted partitions of n. The author announces the following theorems. The numbers $p(49m+k)$, $k=19, 33, 40$; $p(125m+r)$, $r=74, 124$ ($m=0, 1, 2, \cdots$) are respectively divisible by 49 and 125. Proofs are promised in a future paper. *D. H. Lehmer.*

Sprague, R. Über Zerlegungen in n-te Potenzen mit lauter verschiedenen Grundzahlen. Math. Z. 51, 466–468 (1948).

Let $P_n(k)$ denote the number of partitions of k into distinct parts taken from $1, 2^n, 3^n, \cdots$. The author proves that there is a positive integer N_n depending only on n such that $P_n(k)>0$ for all $k>N_n$. In other words, for each n there are only a finite number of integers which are not the sums of distinct nth powers. This result for $n=2$ has already been obtained by the author [same vol., 289–290 (1948); these Rev. 10, 283]. The method of proof is an application of the Tarry-Escott problem. *D. H. Lehmer* (Berkeley, Calif.).

James, R. D. Recent progress in the Goldbach problem. Bull. Amer. Math. Soc. 55, 246–260 (1949).

This paper, recently delivered as an address before the American Mathematical Society, is an expository article. An account is given of Brun's method and Buchstab's modification of it which has led to the result that every sufficiently large even integer is the sum of two numbers each having no more than four prime factors. Schnirelmann's density method and its subsequent improvements are briefly explained; the best result obtained in this direction is that all integers are expressible as the sum of not more than 67 primes. Finally, an account is given of the analytic method which has come closest to solving the problem. This method, due substantially to Hardy and Littlewood, but only successfully exploited by Vinogradov, yields the result that every sufficiently large odd number is the sum of three primes. The bibliography lists 23 papers.

all sufficiently large integers

L. Schoenfeld (Urbana, Ill.).

Errera, A. Sur la démonstration de MM. Artin et Scherk du théorème de M. Mann. Mathematica, Timişoara 23, 70–75 (1948).

The author gives a new arrangement of the proof of the fundamental theorem on the density of sums of sets of positive integers. The proof is based on a rearrangement of the reviewer's original proof due to Artin and Scherk [Ann. of Math. (2) 44, 138–142 (1943); these Rev. 4, 212].

H. B. Mann (Columbus, Ohio).

Hua, L. K., and Vandiver, H. S. Characters over certain types of rings with applications to the theory of equations in a finite field. Proc. Nat. Acad. Sci. U. S. A. 35, 94–99 (1949).

The authors consider the equation

$$c_1x^{a_1}+\cdots+c_sx^{a_s}+c_{s+1}=0,$$

where the coefficients c_i and the unknowns x_i are elements of a finite field $F(p^n)$ with $c_1c_2\cdots c_sx_1\cdots x_s\neq 0$. The exponents a_i are integers such that $0<a_i<p^n-1$; furthermore, $s\geqq 2$ for $c_{s+1}\neq 0$ and $s>2$ for $c_{s+1}=0$. The number of solutions N of this equation is expressed in terms of the characters of the group formed by the nonzero elements of $F(p^n)$ by means of certain generalized Lagrange resolvents. From known theorems about these characters it is then shown that $N=p^{-n}(p^n-1)^s+O(p^{n(s-v)/2})$, where $v=1$ or 0 according as $c_{s+1}\neq 0$ or $c_{s+1}=0$.

H. W. Brinkmann.

Peck, L. G. Diophantine equations in algebraic number fields. Amer. J. Math. 71, 387–402 (1949).

Soient K un corps (de degré fini) de nombres algébriques, $\mathfrak{J}$ son domaine d'intégrité, n son degré. L'auteur prouve les résultats suivants: (I) $f_i(x_1, x_2, \cdots, x_q)=0$ $(i=1, 2, \cdots, h)$ étant un système d'équations en $x_1, x_2, \cdots, x_q$, où f_i est un polynome homogène d'un degré m_i à coefficients dans $\mathfrak{J}$, si K est totalement complexe (c'est-à-dire n'a aucun corps conjugué réel), et si m est un entier naturel, il existe un nombre positif $Q=Q(m; m_1, \cdots, m_h; n)$ tel que, si $q\geqq Q$, le système $f_i=0$ a m solutions dans $\mathfrak{J}$ qui sont linéairement indépendentes; (II) si $\alpha_1, \alpha_2, \cdots, \alpha_q\in\mathfrak{J}$, si la forme $f=\alpha_1z_1^m+\alpha_2z_2^m+\cdots+\alpha_qz_q^m$ est indéfinie (c'est-à-dire, dans tout corps conjugué réel K' de K, la forme conjuguée $f'=\alpha_1'z_1^m+\alpha_2'z_2^m+\cdots+\alpha_q'z_q^m$ de f peut s'annuler pour des valeurs réelles non toutes nulles des $z_1, z_2, \cdots, z_q$) et si $q\geqq 1+\max[4m^{2n+3}, (2^{m-1}+n)mn]$, l'équation $f=0$ a des solutions non-triviales dans $\mathfrak{J}$. Le résultat (I) se trouve être, en vertu d'un résultat de R. Brauer [Bull. Amer. Math. Soc. 51, 749–755 (1945); ces Rev. 7, 108], une conséquence de (II). L'auteur démontre (II) par des méthodes apparentées à celles de Siegel [Ann. of Math. (2) 46, 313–339 (1945); ces Rev. 7, 49] qui consistent à mettre sous forme d'une intégrale le nombre de solutions de $f=0$ dans $\mathfrak{J}$ telles que tous les conjugués des valeurs des $z_1, z_2, \cdots, z_q$ ne dépassent pas, en valeur absolue, un nombre positif T, et de montrer que, quand $T\to+\infty$, cette intégrale devient positive. Pour le faire, le domaine d'intégration est décomposé en deux parties, dont une est la réunion des voisinages convenables $B(\gamma)$ des éléments γ de K, parcourant un système complet de restes modulo δ^{-1}, où δ est la différente de K, dans le sous-ensemble de K, formé par ses éléments dont la norme est bornée, d'une manière convenable, en fonction des m, n, α_i $(i=1, 2, \cdots, q)$ et T. Il est prouvé que la contribution à l'intégrale de cette partie est, à $o(T^{n(q-m)})$ près, de la forme $\sigma(T)I(1)T^{n(q-m)}$, où $I(1)$ est un certain intégrale multiple, et où $\sigma(T)$ est une certaine somme finie, et que la contribution de la partie restante est $o(T^{n(q-m)})$. Ensuite, il est montré que $I(1)>0$ (méthode surtout analytique) et que $\sigma(T)>0$ (méthode surtout algébrique).

M. Krasner (Paris).

ANALYSIS

Stečkin, S. B. On positive bilinear forms. Doklady Akad. Nauk SSSR (N.S.) 65, 17–20 (1949). (Russian)

The bilinear form $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m+n)^{-\lambda}x_my_n$ is bounded in $[p, q]$ if $p^{-1}+q^{-1}\geqq 1$, $\lambda=p'^{-1}+q'^{-1}$, $p'=p/(p-1)$, $q'=q/(q-1)$; the case $\lambda=1$, $p=2$ is Hilbert's inequality [see Hardy, Littlewood and Pólya, Inequalities, Cambridge University Press, 1934, § 9.14; the bound $\{\pi\csc(\pi/(\lambda p'))\}^{\lambda}$ was given by V. Levin, J. Indian Math. Soc. (N.S.) 2, 111–115 (1936)]. Here the author shows that the result is actually a consequence of the case $\lambda=1$ and that Levin's bound is exact. This is a special case of the following theorem on bilinear forms with nonnegative coefficients a_{mn}: if $\sum\sum a_{mn}x_my_n$ has bound $K(p)$ in $[p, p']$, $p>1$, and q and λ are defined as above, then $\sum\sum a_{mn}^{\lambda}x_my_n$ has bound $K(\lambda q')^{\lambda}$ in $[p, q]$. This depends on the following generalization of M. Riesz's convexity theorem for forms with nonnegative coefficients. Let $\sum\sum a_{mn}x_my_n$ and $\sum\sum b_{mn}x_my_n$ be two such forms with respective bounds K_1 and K_2 in $[1/\alpha_1, 1/\beta_1]$ and $[1/\alpha_2, 1/\beta_2]$, and let $0<\alpha_j\leqq 1$, $0<\beta_j\leqq 1$, $0<k_j<1$ $(j=1, 2)$; $k_1+k_2=1$, $\alpha=k_1\alpha_1+k_2\alpha_2$, $\beta=k_1\beta_1+k_2\beta_2$. Then $\sum\sum a_{mn}^{k_1}b_{mn}^{k_2}x_my_n$ is bounded in $[1/\alpha, 1/\beta]$ with bound $K_1^{k_1}K_2^{k_2}$. The generalization follows from Hölder's inequality in the same way as Riesz's case $a_{mn}=b_{mn}$ [see Hardy, Littlewood and Pólya, op. cit., theorem 285]. The theorem follows on taking $b_{mn}=1$.

Finally the author states the following analogue of his lemma: let $a_{mn}(t)$, $\alpha(t)$, $\beta(t)$, $k(t)$ be nonnegative functions, $0\leqq t\leqq 1$, $\int_0^1k(t)dt=1$, $\alpha=\int_0^1k(t)\alpha(t)dt$, $\beta=\int_0^1k(t)\beta(t)dt$; if $\sum\sum a_{mn}(t)x_my_n$ is, for each t, bounded in $[1/\alpha(t), 1/\beta(t)]$ with bound $K(t)$, then $\sum\sum\mathfrak{G}(a_{mn})x_my_n$ is bounded in $[1/\alpha, 1/\beta]$ with bound $\mathfrak{G}(K)$, where $\mathfrak{G}(f)=\exp\{\int_0^1k(t)\log f(t)dt\}$.

R. P. Boas, Jr. (Providence, R. I.).

Valiron, G. Remarques sur un théorème de S. Bernstein. Revista Unión Mat. Argentina 13, 141–144 (1948).

The main result of this note is a proof of the fact that if $f(x)$ is infinitely differentiable on $-1\leqq x\leqq 1$ and $f^{(2n)}(x)\geqq 0$,

The statement (line 8-9 of the review) that the author asserts shows that Levin's bound is exact is wrong. Throughout the review, read "a bound" instead of "bound"; it is not shown that the bounds are exact.

(Errata, these Rev. 11, 870)

$n=0, 1, \cdots$, then $f(x)$ can be continued into the circle $|z|<1$. The author is apparently unaware that this result was announced by I. J. Schoenberg [Bull. Amer. Math. Soc. 47, 389 (1941)] who also states that the result is best possible. *H. Pollard* (Ithaca, N. Y.).

Dugué, Daniel. Sur certaines conséquences qu'entraîne pour une série de Fourier le fait d'avoir tous ses coefficients positifs. Complément au théorème de Weierstrass. C. R. Acad. Sci. Paris **228**, 1469–1470 (1949).

The author states without detailed proof that a function $f(x)$ which is continuous and even in $-\pi \leqq x \leqq \pi$, analytic at $x=0$ and with all its Fourier coefficients positive, is analytic everywhere. He also states that a function which is the limit of a sequence of polynomials with positive coefficients on $(-1, 1)$ is necessarily analytic, and the same is true if polynomials are replaced by finite linear combinations of x^{λ_n}, where λ_n are not integers.

R. P. Boas, Jr. (Providence, R. I.).

Obrechkoff, N. Sur les solutions bornées de quelques équations intégrales singulières. C. R. Acad. Bulgare Sci. Math. Nat. **1**, no. 2–3, 1–4 (1948).

The main theorem was given in C. R. Acad. Sci. Paris **224**, 993–995 (1947); these Rev. **8**, 448. Several special cases are also given here. *R. P. Boas, Jr.*

Sargent, W. L. C. On fractional integrals of a function integrable in the Cesàro-Perron sense. Proc. London Math. Soc. (2) **51**, 46–80 (1949).

Let $F_\alpha(x) = \{\Gamma(\alpha)\}^{-1}\int_a^x (x-t)^{\alpha-1}f(t)dt$, $\alpha>0$, be the Riemann-Liouville integral of order α of $f(x)$, $a \leqq x \leqq b$. The author extends to functions f integrable in the Cesàro-Perron sense the results familiar for $f \in L$, namely: (1) if f is integrable, then $F_\alpha(x)$ exists almost everywhere in x if $\alpha>0$, and everywhere if $\alpha \geqq 1$;

$$(2) \qquad F_{\alpha+\beta}(x) = \{\Gamma(\beta)\}^{-1}\int_a^x (x-t)^{\beta-1}F_\alpha(t)dt.$$

[See also same Proc. (2) **50**, 330–348 (1948); these Rev. **10**, 187.] *A. Zygmund* (Chicago, Ill.).

Calculus

***Humbert, Pierre, et Colombo, Serge. Introduction Mathématique à l'Étude des Théories Électromagnétiques. Fascicule I. Analyse Vectorielle, Transformation Conforme, Théorie du Potentiel.** Gauthier-Villars, Paris, 1949. iv+149 pp.

This book deals with mathematical topics useful to engineers and physicists in their study of the theory of radio and micro-waves. The topics include vector analysis and Maxwell's equations, complex variables, special functions, Laplace's equation and other equations related to it. The discussions are clear but make free use of results presumably familiar to the reader from his course in general mathematics. *P. Franklin* (Cambridge, Mass.).

***Tricomi, Francesco. Esercizi e Complementi di Analisi Matematica. Parte Seconda.** CEDAM, Padova, 1949. vii+376 pp. 1600 lire.

This supplements the author's Lezioni di Analisi Matematica, parte seconda [4th ed. 1939; these Rev. **1**, 298].

***Gröbner, Wolfgang, und Hofreiter, Nikolaus. Integraltafel. Erster Teil. Unbestimmte Integrale.** Springer-Verlag, Vienna, 1949. viii+166 pp. $5.40.

In addition to the types of integrals usually found in short tables, there are extensive sets of elliptic integrals and integrals of combinations of hyperbolic functions. No attempt is made (wisely, in the reviewer's opinion) to use principal values of multiple valued functions, the point of view being adopted that formulas are in general to represent analytic functions of a complex variable.

R. P. Boas, Jr. (Providence, R. I.).

Wilkins, J. Ernest, Jr. An integration scheme of Maréchal. Bull. Amer. Math. Soc. **55**, 191–192 (1949).

An approximation procedure of Maréchal [J. Opt. Soc. Amer. **37**, 403–404 (1947)] is justified by establishing that $\lim_{a\to 0} 2\pi a\int_C f(r,\theta)ds = \int_0^{2\pi}\int_0^R f(r,\theta)rdrd\theta$, where $f(r,\theta)$ has period 2π in θ and is continuous in the region $0 \leqq r \leqq R$, $0 \leqq \theta < 2\pi$ and C is that part of the curve $r=a\theta$ in the region.

P. Civin (Eugene, Ore.).

Ryšavý, Vladimír. Two elliptic cubatures. Časopis Pěst. Mat. Fys. **73**, D52–D56 (1949). (Czech)

The author studies two surfaces related to the ellipse E: $x^2/a^2+y^2/b^2-1=z=0$. The one is the envelope of all spheres whose centres are on E and which pass through the origin; the other is the locus of circles in planes through the z axis whose centre is at the origin and which meet E. He shows that the volume enclosed by either of these surfaces can be expressed in terms of elliptic integrals.

A. Erdélyi (Pasadena, Calif.).

Hummel, P. M., and Seebeck, C. L., Jr. A generalization of Taylor's expansion. Amer. Math. Monthly **56**, 243–247 (1949).

By repeated integration by parts, starting from

$$\int_a^x f^{(m+n+1)}(t)(x-t)^m(t-a)^n dt,$$

the authors show that, if $f(x)$ has enough derivatives,

$$f(x)-f(a) = \sum_{k\geq 1} \frac{(m+n-k)!}{(m+n)!} \times \{\binom{m}{k}f^{(k)}(a) - (-1)^k\binom{n}{k}f^{(k)}(x)\}(x-a)^k + R,$$

where $\binom{p}{k}=0$ if $k>p$ and

$$R = (-1)^n \frac{m!n!(x-a)^{m+n+1}}{(m+n)!(m+n+1)!} f^{(m+n+1)}(\theta),$$

with θ between a and x. This includes as a special case a formula useful in the calculation of nth roots, discussed by V. A. Bailey, J. S. Frame and H. S. Wall [see Wall, same Monthly **55**, 90–94 (1948); these Rev. **9**, 382]. The application of the case $f(x)=e^x$ to the calculation of e^a is also discussed. [The authors were apparently unaware that their formula is a special case of a formula of Darboux; see Whittaker and Watson, A Course of Modern Analysis, § 7.1.] *R. P. Boas, Jr.* (Providence, R. I.).

***Schmidt, Friedrich Karl. Vektorrechnung. Teil II.** Aschendorff'sche Verlagsbuchhandlung, Münster, 1948. iii+244 pp. 6 Marks.

This work contains a logical and reasonably rigourous development of vector analysis for n-space and function space. Careful consideration is given to limiting processes

and the whole is illustrated by applications to geometry and physical science. The contents are as follows: (I) Euclidean vector space, linear metric vector spaces, linear topological vector spaces; (II) Differentiation of functions which depend on a parameter; (III) Differentiation of vector functions; (IV) Differentiation of operators; (V) Integral theorems in Euclidean vector space. The last chapter gives a useful discussion of the generalised form of Stokes's theorem in a form which can be extended to orientable manifolds which can be mapped on a simplex. The author's method, based on "σ-forms" and "boundary construction", appears to be new and is certainly more intuitive than the usual method, based upon "differential forms" and "differentiation of a differential form." In the author's notation the general theorem is enunciated as

$$\int_{\text{Boundary of } B} \varphi = \int_B \text{Boundary of } \varphi,$$

where this way of writing the integral has a properly defined meaning. The ordinary theorems of Stokes and Gauss are particular cases of the above.

L. M. Milne-Thomson (Greenwich).

Bilimovitch, Anton. Aires et volumes vélocidiques et hodographiques dans un mouvement du fluide. Acad. Serbe Sci. Publ. Inst. Math. **2**, 37–52 (1948). (French. Serbian summary)

Let v be any plane solenoidal vector field, let A_v be the area inclosed by an arbitrary closed contour, let A_r be the area inclosed by the curve consisting of the termini of the vectors v at each point of the curve, and let A_H be the area inclosed by the curve consisting of the termini of these same vectors when the initial point of each is put at the origin. D. Pompeiu [Bull. Math. Phys. Éc. Polytech. Bucarest **1**, 42–43 (1929)] proved that $A_r = A_v + A_H$. The author first gives a vectorial proof of this result. An essential lemma is the fact that from $\operatorname{div}_r v = 0$ it follows that $\operatorname{div}_v r = 0$, for a plane field. This lemma does not hold for the three-dimensional case unless $II_d + w^2 = 0$, where II_d is the second principal invariant of the rate of deformation tensor and w is the vorticity. Thus the author proves that Pompeiu's result can be extended to volumes in a three-dimensional solenoidal field subject only to the above condition. *C. Truesdell* (Washington, D. C.).

Theory of Sets, Theory of Functions of Real Variables

Levi, Beppo. Historical and critical essay on the arithmetic of sets and the continuum problem. Math. Notae **8**, 6–78 (1948). (Spanish)

The author gives an account, in part historical, of certain aspects of the theory of transfinite cardinals and ordinals. An "axiomatic" definition of ordinals and the sequence of alephs is preferred to the "constructive" approach of Cantor. Addition, multiplication and exponentiation are defined and many of the usual theorems are proved. Several of these are concerned with the nature of the conditions $2a = a$ or $a^2 = a$, where a is an arbitrary cardinal number; this is in part a commentary on a recent paper of Denjoy [C. R. Acad. Sci. Paris **224**, 1080–1083 (1947); these Rev. **8**, 505]. A detailed discussion is given of the sequence $c_1 = 2^{\aleph_0}$, $c_2 = 2^{c_1}$, $\cdots$, and the sequence $\aleph_0, \aleph_1, \cdots$, and their relations, with and without the well ordering axiom or the assumption of comparability. The author doubts the validity of Sierpiński's proof [Leçons sur les Nombres Transfinis, Gauthier-Villars, Paris, 1928, p. 210] that $\aleph_1 < c_2$. *R. C. Buck.*

Fraïssé, Roland. Sur la comparaison des types de relations. C. R. Acad. Sci. Paris **226**, 987–988 (1948).

Let B be a set. Then a subset R of B^2 is a relation with base B. Two relations R and R' are isomorphic if there is a one-to-one correspondence between their bases which preserves the relation. This isomorphism dissects the class of all relations into classes, called types, $T, T', \cdots$. Write $T < T'$ if there are R and R' in T and T', respectively, such that R can be obtained from R' by diminishing the base of R', and write $T \ll T'$ if, in addition, T' is not $< T$, and write $T' \approx T$ if, in addition, $T' < T$. The following results are announced. (1) There are four types such that any denumerable type is $>$ one of these, a denumerable type being one containing relations on denumerable base. (2) There are denumerable types each of which is $>$ all denumerable types. This property, called richness, is preserved when a point in the base is deleted. (3) Given a sequence of nonrich types, there is a nonrich type $\gg$ all of these. There are also results on binary decomposition of types. *R. Arens.*

Fraïssé, Roland. Sur la comparaison des types d'ordres. C. R. Acad. Sci. Paris **226**, 1330–1331 (1948).

Ordered sets being defined by relations, the terms and results of the note reviewed above apply here. There are stated additional results on order types, such as characterization of rich types and indecomposable types.

R. Arens (Los Angeles, Calif.).

Tola Pasquel, José. On a certain generalization of the notion of equivalence. Revista Ci., Lima **50**, 205–213 (1948). (Spanish)

Given a set C, suppose for each $\mu = 0, 1, \cdots, m$, F_μ is a family of subsets ("configurations") $K_\mu^\alpha \subset C$ such that: K_0 is void; if $S \subset C$ has fewer than μ elements then $S \subset K_\mu^\alpha$ for some α; if $S \subset C$ has μ elements and $S \not\subset K_{\mu-1}^\beta$ for all β then $S \subset K_\mu^\alpha$ for exactly one α; for given μ and α, the subset K_μ^α contains μ elements which do not all belong to the same $K_{\mu-1}^\beta$. Define a unary relation R_μ ($\mu = 0, 1, \cdots, m$) on subsets of C: $R_\mu S$ if and only if $S \subset K_\mu^\alpha$ for some α. Then R_0 is void, R_1 is an equivalence relation, and R_μ ($\mu > 1$) somewhat resembles an equivalence relation. A direct set of postulates for the R_μ is also given. *P. M. Whitman.*

Everett, C. J., and Whaples, G. Representations of sequences of sets. Amer. J. Math. **71**, 287–293 (1949).

Soit (M_α), $\alpha \varepsilon A$, une famille de parties non vides d'un ensemble E; on dit que cette famille est représentable s'il existe une application biunivoque $\alpha \to f(\alpha)$ de A dans E telle que $f(\alpha) \varepsilon M_\alpha$ pour tout $\alpha \varepsilon A$. P. Hall a démontré [J. London Math. Soc. **10**, 26–30 (1935)] que si A est un ensemble fini, pour que la famille (M_α) soit représentable, il faut et il suffit que, pour toute partie B de A, l'ensemble $\cup_{\alpha \varepsilon B} M_\alpha$ ait une puissance au moins égale à celle de B. Les auteurs donnent une démonstration nouvelle de ce résultat, montrent qu'il n'est plus vrai sans restriction lorsque A est infini, mais qu'il admet la généralisation partielle suivante: si tous les M_α sont des ensembles finis, pour que (M_α) soit représentable, il faut et il suffit que, pour toute partie finie B de A, l'ensemble $\cup_{\alpha \varepsilon B} M_\alpha$ ait une puissance au moins égale à celle de B. Ils en déduisent une démonstration d'un théorème de de Bruijn [Nieuw Arch. Wiskunde (2) **22**, 48–52 (1943); ces Rev. **7**, 277] sur les représentations communes de deux partitions d'un ensemble. *J. Dieudonné.*

Wagner, Klaus. Charakterisierung wohlgeordneter Mengen und der Zahlengeraden mit Hilfe eines allgemeinen Metrisierbarkeitsbegriffes. Math. Ann. 120, 502–513 (1949).

Let A be an ordered set with a least element, and for $a \in A$ let A_a be the class of $b \geqq a$. Call A an M-set if for $a \in A$ there is an isomorphism $b \to ba \geqq a$ for $b \in A$. Call A an MA-set if $a(cb) = (ac)b$, and call A an MK-set if $(ac)b = (ab)c$. The well-ordered M-, MK-, and MA-sets are characterized. The real numbers and the integers are shown to be the only complete MK- and MA-sets. The MK- and MA-sets which are not subsets of the real numbers are shown to be decomposable in a certain sense. *R. Arens.*

Marczewski, E. Concerning the symmetric difference in the theory of sets and in Boolean algebras. Colloquium Math. 1, 199–202 (1948).

Let $\mathfrak{B}$ be a Boolean algebra with zero element 0 and universal element 1. Consider a binary operation $\circ$ and a subset $\mathfrak{G}$ of $\mathfrak{B}$ possessing the following properties: (1) $\mathfrak{G}$ contains 0 and 1; (2) $\mathfrak{G}$ is a group under the operation $\circ$ with 0 as unit element; (3) the operation $\circ$ is quasi-analytical over $\mathfrak{G}$; that is,

$$(A_1 \circ A_2) \ominus (B_1 \circ B_2) \subset (A_1 \ominus B_1) \cup (A_2 \ominus B_2),$$

where $\ominus$ is the Boolean symmetric difference operation. It is shown that the operations $\circ$ and $\ominus$ coincide over $\mathfrak{G}$. Some analogues between the symmetric difference of sets and the absolute value of the difference of numbers are also mentioned. *W. Gustin* (Bloomington, Ind.).

Helson, Henry. On the symmetric difference of sets as a group operation. Colloquium Math. 1, 203–205 (1948).

Let G be a set and $\mathfrak{G}$ the collection of all subsets of G. Consider a binary set operation $\circ$ possessing the following properties: (1) $\mathfrak{G}$ is a group under the operation $\circ$ with the null set as unit element; (2) every simple mapping is a homomorphism of this group into itself, a simple mapping being a mapping of G onto itself which interchanges some two elements of G and leaves the remaining elements of G fixed; (3) $A \circ B \subset A \cup B$. It is shown that these three properties characterize the binary set operation of symmetric difference over $\mathfrak{G}$. *W. Gustin* (Bloomington, Ind.).

Obreanu, F. Corrections to "La puissance de certaines classes de fonctions." Duke Math. J. 15, 593 (1948).

The paper appeared in the same J. 14, 377–380 (1947); these Rev. 9, 230.

Lyapunov, A. A. On δs-operations preserving measurability and the property of Baire. Mat. Sbornik N.S. 24(66), 119–127 (1949). (Russian)

Let I be any separable metric space which is an absolute Borel set. If Ξ is any family of subsets of I, a set $E \subset I$ is said to be Ξ-measurable if $E = E_1 \cup E_2$, where E_1 is a Borel set and $E_2 \in \Xi$. A family Ξ of subsets of I is called a rarefied family if (1) Ξ is closed under the formation of countable unions; (2) Ξ contains all subsets of every set in Ξ; (3) every $E \in \Xi$ is a subset of a Borel set in Ξ; (4) I non$\in \Xi$; (5) given any family of pairwise disjoint Ξ-measurable sets, all but a countable number of these sets are in Ξ. A subset A of I is said to be descriptively measurable if it is measurable with respect to every rarefied family Ξ. The author proves that the image of a descriptively measurable set under a one-to-one Borel-measurable mapping is again descriptively measurable. An operation T, defined for countable families of sets, and carrying such families into single sets, is said to be Ξ-measurable (descriptively measurable) if $T(\{E_n\})$ is Ξ-measurable (descriptively measurable) whenever all of the sets E_n are Ξ-measurable (descriptively measurable). The author proves that a number of special T-operations are Ξ-measurable and descriptively measurable. He next proves that the number of non-equivalent descriptively measurable T-operations is $2^{\aleph_1}$, and finally gives a partial solution to the problem of representing all descriptively measurable T-operations in terms of certain special ones.

E. Hewitt (Seattle, Wash.).

Lyapunov, A. A. On continuous transformations of A-sets. Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 61–64 (1949). (Russian)

The following theorems are proved. (I) Let E be an analytic subset of the space I of irrational numbers, and let f be a continuous mapping of E into I. Then, for every $\epsilon > 0$, there exists a closed subset F of E such that $0 \leqq \mu(f(E)) - \mu(f(F)) < \epsilon$, where μ is Lebesgue measure, and the mapping f is a homeomorphism on F. (II) If E is an analytic set and f is a continuous mapping of E into I, there exists a set $B \subset E$ of type G_δ such that $f(E) - f(B)$ is of first category and f is a homeomorphism on B.

E. Hewitt (Seattle, Wash.).

Lyapunov, A. A. On set-theoretical operations which preserve measurability. Doklady Akad. Nauk SSSR (N.S.) 65, 609–612 (1949). (Russian)

The author defines a set-theoretical operation with basis N on an arbitrary sequence $\{E_n\}$ of sets by the formula

$$\psi_N(\{E_n\}) = \sum_{\nu \in N} \prod_{n \in \nu} E_n \cdot \prod_{m \in \nu} CE_m.$$

A process is described, by means of which, starting with a set-theoretical operation $\psi_N \equiv \psi_{N_0}$, it is possible to define operations ψ_{N_α} for α in the first, second, and third classes of ordinals. The process is not unique for all ordinals of the third class; this requires the introduction of the term "$\psi_{N_{\alpha_1}}$ agrees with $\psi_{N_{\alpha_2}}$." Denoting by Ξ the class of all sets in a Baire space, the author states (1) if $\alpha_1 < \alpha_2$ and the operation $\psi_{N_{\alpha_1}}$ agrees with $\psi_{N_{\alpha_2}}$, then the class $\psi_{N_{\alpha_1}}(\Xi)$ constitutes the regular part of the class $\psi_{N_{\alpha_2}}(\Xi)$; (2) if the operation ψ_{N_0} preserves measurability, then all the operations ψ_{N_α} do also; (3) if $\psi_{N_0} = \Sigma$, then the class of R-sets [same Doklady (N.S.) 58, 1887–1890 (1947); these Rev. 9, 339] coincides with the sum $\sum_{\alpha < \omega_1} \psi_{N_\alpha}(\Xi)$. *H. L. Smith* (Baton Rouge, La.).

Horn, Alfred, and Tarski, Alfred. Measures in Boolean algebras. Trans. Amer. Math. Soc. 64, 467–497 (1948).

In this paper the authors establish a variety of results concerning the relationship between the structure of a Boolean algebra A and the kinds of finitely additive measures which exist on it. The first section is largely devoted to the problem of extending a nonnegative function $f(x)$ on an arbitrary subset S of A to a measure on A. It is shown that this can be done if f is a "partial measure" on S, that is, if $1 \in S$ and $f(1) = 1$, and if $\sum_1^n f(a_i) \leqq \sum_1^m f(b_j)$ whenever, for every $k \leqq m$, the union of all the products $\bigcap_{i=1}^k a_{i_i}$ over subsequences of $\{a_i\}$ of length k is contained in the corresponding union of all products over subsequences of $\{b_j\}$ of length k. The second section is concerned with strictly positive measures. For example, the authors consider the following conditions: (0) A is separable (that is, A contains

a countable subset B such that if $0 \neq x \in A$ then there is an element $y \in B$ such that $0 \neq y \subseteq x$); (i) there is a strictly positive measure on A; (ii) A can be represented as the union of an infinite sequence of sets none of which includes an infinite subset of pairwise disjoint elements; (iii) every uncountable subset of A includes an uncountable subset in which no two elements are disjoint; (iv) every ramification set of A is countable (see the paper for this notion); (v) every set of pairwise disjoint elements of A is countable. It is shown that (i)$\rightarrow$(ii)$\rightarrow$(iii)$\rightarrow$(iv)$\rightarrow$(v), and that if A has a basis which is linearly ordered under inclusion then (iv)$\rightarrow$(0)$\rightarrow$(i). The final section establishes several partial criteria for the existence of countably additive measures. For instance, it is shown that for A to have a strictly positive countably additive measure it is necessary for A to be weakly countably distributive, and that if A is separable then this condition is also sufficient and is equivalent to A's being atomistic. *L. H. Loomis* (Cambridge, Mass.).

Maharam, Dorothy. The representation of abstract measure functions. Trans. Amer. Math. Soc. **65**, 279–330 (1949).

The author defines an abstract measure algebra as a Boolean σ-algebra satisfying the countable chain condition and possessing as part of its intrinsic structure an equivalence relation connected with the algebraic operations by three natural and simple postulates. A suggestive model for such a structure is the Boolean algebra of measurable sets modulo sets of measure zero in a σ-finite measure space, equivalence being interpreted as having the same measure. Another model is an arbitrary Boolean σ-algebra satisfying the countable chain condition, equivalence this time being interpreted as identity. The main result is that every abstract measure algebra can be obtained from these two models by means of elementary algebraic constructions. More precisely, in the terminology of the author, every abstract measure algebra is isomorphic to a principal ideal in the free direct sum of two direct products, in each of which one factor is numerical and the other trivial. A reformulation of this result asserts that every abstract measure algebra is isomorphic to a Boolean algebra with a measure whose values are nonnegative numerical functions on some set, modulo a suitable ideal of null functions.

The concept of direct product is motivated and in fact defined by the formation of a Boolean algebra of sets in the Cartesian product of two spaces. To obtain the spaces necessary for the formation of direct products in the first formulation and the domain of the functions in the second formulation, the author makes use of Stone's theory of representation of Boolean algebras. Most of the methods of the paper are adaptations of those devices which are used in the proof of set-theoretic results such as the Schroeder-Bernstein theorem, and which have been exploited more recently and in more algebraic form in the proof of the Banach-Tarski paradox and in von Neumann's construction of dimension functions.

The paper concludes with some corollaries and applications. Among these may be mentioned (1) a characterization of those abstract measure algebras which are isomorphic to numerical measure algebras, (2) a subsumption of the author's classification of numerical measure algebras under the present theory [cf. Proc. Nat. Acad. Sci. U. S. A. **28**, 108–111 (1942); these Rev. **4**, 12], and (3) a decomposition theorem for measure-preserving automorphisms of a numerical measure algebra. *P. R. Halmos* (Chicago, Ill.).

Dieudonné, Jean. Sur le théorème de Lebesgue-Nikodym. III. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **23**, 25–53 (1948).

[For parts I and II cf. Ann. of Math. (2) **42**, 547–555 (1941); Bull. Soc. Math. France **72**, 193–239 (1944); these Rev. **3**, 50; **7**, 305.] Two further generalizations of the Lebesgue-Nikodym theorem are given, which involve functions with values in a topological vector space F over the real number field, one of them for integrals including those of Gelfand-Dunford, the other integrals of Bochner. Proofs are clearly presented by interplay between a space E (with measure function μ) and an associated space $\bar{E}$ of Kakutani. A result of Halmos about quotient measures (although correctly deduced from a lemma stated by Doob, it needs stronger hypotheses as shown by example) is established and furnishes a basis for the second part. Without attempting the definitions involved, suffice it to say that for f vectorial-valued and g real-valued, relations are discussed between continuous linear operators and $\int g f d\mu$ (values in F) as a function of f in one case, of g in the other.

J. F. Randolph (Rochester, N. Y.).

Lorentz, G. G. A problem of plane measure. Amer. J. Math. **71**, 417–426 (1949).

A pair of functions $P(x)$, $Q(y)$ is termed the pair of cross functions for a plane set A if, for almost every x_0 and almost every y_0, the sections of A by the line $x=x_0$ and the line $y=y_0$ have, respectively, the linear measures $P(x_0)$, $Q(y_0)$. It is proved that in order that two given nonnegative integrable functions $P(x)$, $Q(y)$ defined for all real x, y be a pair of cross functions for at least one measurable set A of finite plane measure, it is necessary and sufficient that their nonincreasing rearrangements $p(x)$, $q(x)$ defined for nonnegative x, y satisfy the two conditions

$$\int_0^x p(u)du \leqq \int_0^x q^{-1}(u)du, \quad \int_0^x q(u)du \leqq \int_0^y p^{-1}(u)du,$$

and that, moreover, the set A is uniquely determined modulo null sets if and only if p, q are inverse functions. The author remarks further that quite a different situation prevails if A is not restricted to be of finite plane measure.

L. C. Young (Madison, Wis.).

Rodríguez-Salinas, Baltasar. The inversion of the order of differentiation. Revista Acad. Ci. Madrid **42**, 37–70 (1948). (Spanish)

The author presents a comparative study of various criteria for a function $f(x, y)$ to have at (x_0, y_0) equal second-order partial derivatives $f_{xy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$. Related questions are discussed, including, in particular, results on fractional derivatives and the differentiation of certain types of functionals. *W. T. Reid* (Evanston, Ill.).

Zwirner, G. Alcuni teoremi di geometria infinitesimale diretta relativi alle curve spaziali. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 113–116 (1948).

Let $x=x(t)$, $y=y(t)$ $(a \leqq t < b)$ be equations of a simple curve in the (x, y)-plane whose length from a to t is finite for each t, and let $f(z)=f(x+iy)$ be continuous in t when $x=x(t)$, $y=y(t)$. If, for each t, $\lim_{\Delta t \to 0} \Delta f[z(t)]/\Delta z(t)=0$, then $f(z)$ is constant on the curve. This is proved using the equation $\Delta f/\Delta s = |\Delta f/\Delta z| \cdot |\Delta z/\Delta s|$, where s is arc length. The hypothesis of finite length can be replaced by the hypothesis that a unique tangent exists at each point. This result is shown to follow from an earlier theorem of the

author [same Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 524–530 (1948); these Rev. 10, 265], namely that a curve in 3-space having a tangent at each point, always parallel to a given plane, is in a plane parallel to that plane. The condition that the curve is simple can be removed by proper wording of the theorems. *A. B. Brown* (Flushing, N. Y.).

Mařík, Jan. Estimate of mean values of integrals and criteria for the convergence of improper integrals. Časopis Pěst. Mat. Fys. 73, D49–D52 (1949). (Czech)

Let f and g be real functions, f integrable and g of bounded variation V on $a \leqq x \leqq b$. The author proves the existence of a number ξ, $a \leqq \xi \leqq b$, such that

$$\left| \int_a^b fg\,dx \right| \leqq \{ V + |g(b)| \} \left| \int_a^\xi f\,dx \right|,$$

and uses this inequality to prove sufficient conditions for the existence of the (possibly infinite or improper) integral $\int_a^b fg\,dx$. These are: for every c, $a < c < b$, f is properly Riemann integrable over (a, c) and the total variation of g in (a, c) is less than B (independent of c), and either $\lim_{c \to b-} \int_a^c f\,dx$ exists or $\left| \int_a^c f\,dx \right| < A$ for all c, and $\lim_{x \to b-} g(x) = 0$. *A. Erdélyi* (Pasadena, Calif.).

Ingham, A. E. Improper integrals as limits of sums. J. London Math. Soc. 24, 44–50 (1949).

Let $\lambda_1 < \lambda_2 < \cdots$ and $0 = l_0 < l_1 < \cdots$ be unbounded sequences such that $l_{n-1} \leqq \lambda_n \leqq l_n$ and $d_n = l_n - l_{n-1} = o(l_n)$ as $n \to \infty$. Let K be the class of functions $\varphi(t)$ on $0 < t \leqq 1$ which are bounded and integrable on every interval $(0 <) \ \delta \leqq t \leqq 1$. If $0 < h < 1$, put $S(h) = \sum_{l_n h \leqq 1} \varphi(\lambda_n h) d_n h$ and $I(h) = \int_h^1 \varphi(t)\,dt$. A necessary and sufficient condition that, for functions φ in K, the existence of $S = \lim_{h \to 0} S(h)$ should imply the existence of $I = \lim_{h \to 0} I(h)$, in which case $I = S$, is that $\int^\infty v^{-1} |\gamma(v)|\,dv < \infty$, where $\gamma(v) = \sum_{\rho_n \leqq v} \mu_n / \rho_n$ and μ_n, ρ_n are the numbers occurring in a Dirichlet series $\sum \mu_n / \rho_n^s$ which is formally the reciprocal of $\sum d_n / \lambda_n^s$. This theorem makes clear the relationship of prime number theory to the known result [Ingham, same J. 20, 171–180 (1945); these Rev. 8, 147; Wintner, Amer. J. Math. 69, 685–708 (1947); these Rev. 9, 279, where it is not assumed that φ is bounded] that the existence of S implies that of I when $l_n = \lambda_n = n$. *P. Hartman* (Baltimore, Md.).

Kober, H. On singular functions of bounded variation. J. London Math. Soc. 23, 222–229 (1948).

On an interval $(0, a)$ a function f of bounded variation V is said to be singular if (A) $f'(t) = 0$ for almost all t on $(0, a)$, or (B) given $\epsilon > 0$, there exists a finite number of nonoverlapping intervals (t_k, t_k') in $(0, a)$ such that $\sum (t_k' - t_k) < \epsilon$ and $\sum |f(t_k') - f(t_k)| > V - \epsilon$, or (C) the length of the arc of the curve $y = f(t)$, joining the points $(0, f(0))$, $(a, b(a))$ is equal to $(a + V)$. The equivalence of (C) with the known definitions (A) and (B) is established [and may also be obtained as a direct consequence of results of A. P. Morse, Trans. Amer. Math. Soc. 41, 48–83 (1937), in particular, p. 70]. Also discussed are inverses of singular functions, jump functions, and limits of sequences of singular functions. *J. F. Randolph* (Rochester, N. Y.).

Stark, Richard H. On the representation of a function as a Hellinger integral. Bull. Amer. Math. Soc. 55, 155–159 (1949).

Il s'agit d'un théorème de Hellinger [dissertation, Göttingen, 1907] qui peut s'anoncer ainsi: étant données deux fonctions continues non décroissantes $g(x)$ et $h(x)$ dans un intervalle I, pour qu'il existe une fonction $f(x)$ telle que $h(x) = \int_a^x (df)^2 / dg$, il faut et il suffit que toute partie de I de mesure nulle par rapport à g soit aussi de mesure nulle par rapport à h. L'auteur en donne une démonstration "plus simple que celle de Hellinger, et indépendante de la théorie de Lebesgue." *R. de Possel* (Alger).

Ellis, H. W. Mean-continuous integrals. Canadian J. Math. 1, 113–124 (1949).

A function $F(x)$ is M_r-continuous, $r = 1, 2, \cdots$, at a point a if $r(b-t)^{r-1} F(t)$ is GM_{r-1}-integrable (see below) on (a, b) for b sufficiently near a, and if

$$\lim_{b \to a} M(b-a)^{-r} \int_a^b (b-t)^{r-1} F(t)\,dt = F(a).$$

A function $f(x)$ is GM_r-integrable on an interval $[l, m]$ if there exists a function that is M_r-continuous and ACG on $[l, m]$ (where the definition of ACG implies that the sets over which $F(x)$ is AC are closed), and if $ADF(x) = f(x)$ almost everywhere on $[a, b]$. For $r = 1, 2, \cdots$, this gives a scale of integrals of increasing generality for which the only concept of the descriptive definition of the general Denjoy integral that is generalized is continuity. The GM_r-integral has many of the properties of the general Denjoy integral including a theorem on integration by parts. It contains the corresponding rth order Cesàro-Perron integral of J. C. Burkill [Proc. London Math. Soc. (2) 34, 314–322 (1932)]. A constructive definition of the GM_r-integral is given and shown to be equivalent to the descriptive definition. An indefinite GM_r-integral takes all values between its upper and lower bounds on any interval over which the integral exists. *R. L. Jeffery* (Kingston, Ont.).

Mulholland, H. P. On the total variation of a function of two variables. Corrigendum. Proc. London Math. Soc. (2) 50, 559–560 (1949).

Formula (8.4) of chapter 5 of Saks' "Theory of the Integral" [Warsaw, 1937] is valid only under certain restrictions [L. C. Young, Duke Math. J. 11, 43–57 (1944); these Rev. 6, 121]. This error requires slight changes of formulation in the author's paper [same Proc. (2) 46, 290–311 (1940); these Rev. 1, 303]. *W. Feller.*

Young, L. C. On area and length. Fund. Math. 35, 275–302 (1948).

The author extends his previous results on the isoperimetric ratio of a harmonic map of the disc [Proc. London Math. Soc. (2) 49, 396–408 (1947); these Rev. 9, 459] to maps of arbitrary simply connected regions, defining the length of the bounding curve in terms of prime ends. *H. Federer* (Providence, R. I.).

Radó, Tibor. On the problem of Geöcze. Ann. Scuola Norm. Super. Pisa (2) 14 (1945), 21–30 (1948).

The author proves a refinement of a recent result of Mambriani [same Ann. (2) 13 (1944), 1–17 (1948); these Rev. 9, 417], who solved the problem of Geöcze for surfaces represented by functions which are absolutely continuous in the sense of Tonelli and whose partial derivatives are square integrable. *H. Federer* (Providence, R. I.).

Besicovitch, A. S. On surfaces of minimum area. Proc. Cambridge Philos. Soc. 44, 313–334 (1948).

The author defines $a(f)$, the area of the mapping f, as the integral with respect to two-dimensional Hausdorff measure, over the range of f, of the function which relates

to each point in the range of f the number (possibly ∞) of components of its counter-image under f. We observe that, in case the domain of f is compact, $a(f)$ coincides with $B^2(f)$ introduced by Busemann [Ann. of Math. (2) **48**, 234–267 (1947); these Rev. **8**, 573; we are using the notation of the review]. The author is concerned with the case in which f maps the closed unit disc U into Euclidean three-space.

Let J be the bounding circle of U, and let the Fréchet class of $(f|J)$ be prescribed, the "given closed contour." The author finds that $a(f)$ assumes its greatest lower bound on each such class of maps f, at least in case the prescribed bounding curve is simple. He uses a smoothing process which decreases what would commonly be called the "monotonic deficiency in the sense of Lebesgue," that is, which removes bulges across planes. [The proof of lemma 4, on which this process rests, is omitted, and is not clear to the reviewer.] The author constructs from a minimizing sequence a limit map on an intricate limit space, which is finally shown to be homeomorphic with the disc. The discussion of the case in which the bounding curve is not simple is not carried out specifically.

It occurs to the reviewer that the result of this paper can be derived from known results about Lebesgue area and about the relationship of Lebesgue area to the area a, at least in case the Fréchet class of $(f|J)$ is that of a simple closed curve. For each point y in the range of f let $u_f(y)$ be the number (possibly ∞) of those components of the f counter-image of y which are contained in the interior of U. Let H be the two-dimensional Hausdorff measure. Then $a(f)=\int u_f dH+H(f(J))$. Since $H(f(J))$ is constant on the Fréchet class of $(f|J)$, the problem of minimizing $a(f)$ is equivalent to the problem of minimizing $\int u_f dH$. It is known that $\int u_f dH \geqq L(f)$, the Lebesgue area of f [Cesari, Ann. Scuola Norm. Super. Pisa (2) **10**, 253–295 (1941); **11**, 1–42 (1942); Federer, Trans. Amer. Math. Soc. **62**, 114–192 (1947); **59**, 441–466 (1946); Radó, Proc. Nat. Acad. Sci. U. S. A. **31**, 102–106 (1945); these Rev. **8**, 257; **9**, 231; **7**, 422; **6**, 204]. The precise conditions for equality are also known. It holds, in particular, if f is differentiable on the interior of U. The solution of the Plateau problem for Lebesgue area [Douglas, Trans. Amer. Math. Soc. **33**, 263–321 (1931); McShane, ibid. **35**, 716–733 (1933); Radó, Math. Z. **32**, 763–796 (1930)] states that $L(f)$ is minimized by a map f whose components are harmonic on the interior of U. For this particular map f we have $\int u_f dH=L(f)$. Hence any surface which minimizes the Lebesgue area $L(f)$ also minimizes $\int u_f dH$, and therefore $a(f)$.

The preceding method, which shows that every differentiable solution of the problem of Plateau for Lebesgue area is also a solution of the problem studied by Besicovitch, may be extended from the special case of a simple boundary curve to the case in which the set all multiple points of the boundary curve (points whose counter-image on J is disconnected) has two-dimensional Hausdorff measure zero.

H. Federer (Providence, R. I.).

Besicovitch, A. S. Parametric surfaces. I. Compactness. Proc. Cambridge Philos. Soc. **45**, 5–13 (1949).

For each map f of the closed disc U into three-space let $\Phi(f)$ be the set of all maximal continua of constancy of f, and metrize $\Phi(f)$ by the function d_f such that, for $P, Q \in \Phi(f)$, $d_f(P, Q)$ is the infimum of diameter $(f(C))$ for all continua C which are contained in U and meet both P and Q. For each $\epsilon>0$ let $N(f, \epsilon)$ be the supremum of the cardinal numbers of all sets $S \subset \Phi(f)$ for which $d_f(P, Q) \geqq \epsilon$ whenever P and Q are distinct elements of S. The main result is as follows. A set A of maps of U into three-space is conditionally compact with respect to the Fréchet distance if and only if, for each $\epsilon>0$, the supremum of $N(f, \epsilon)$ for $f \in A$ is finite.

H. Federer (Providence, R. I.).

Besicovitch, A. S. Parametric surfaces. II. Lower semi-continuity of the area. Proc. Cambridge Philos. Soc. **45**, 14–23 (1949).

Let H^* be the measure over three-space which is defined like two-dimensional Hausdorff measure except that only spheres are allowed as covering sets. The author now uses the area $a^*(f)=\int u_f dH^*$ [in the notation of the two preceding reviews]. The interior points of $\Phi(f)$ are those which do not meet J. The main result of this paper is that a^* is lower semi-continuous at a point f in case f is a saddle surface and in case f has a tangent plane, in a very strict sense too complicated to describe here, at H^* almost all interior points.

The reviewer observes that, since $a^*(f) \geqq L(f)$ for every surface f, with equality for sufficiently smooth surfaces, we have $a^*(f)=L(f)$ if and only if a^* is lower semicontinuous at f. Hence the results of this paper may be interpreted as statements to the effect that for certain surfaces the area a^* agrees with the Lebesgue area. In view of known results this implies that these surfaces, in particular saddle surfaces, have very good tangential properties. It also turns out that for such surfaces H and H^* give the same area, that is, $\int u_f dH=\int u_f dH^*$.

H. Federer (Providence, R. I.).

Besicovitch, A. S. Parametric surfaces. III. On surfaces of minimum area. J. London Math. Soc. **23**, 241–246 (1948).

The author simplifies his previous discussion of this problem by applying the results of the two papers reviewed above.

H. Federer (Providence, R. I.).

Youngs, J. W. T. Lebesgue, Fréchet and Kerékjártó varieties. Amer. J. Math. **70**, 481–496 (1948).

Suppose X is a Peano space, Y is metrized by ρ, and the set C of all continuous maps of X into Y is metrized by the function d such that $d(a, b)=\sup \rho[a(x), b(x)]$ for $x \in X$ when $a, b \in C$. The author studies some equivalence relations over C which have been used in the theory of area. Deviating in detail from the author's discussion, we observe with N. Bourbaki [Topologie générale, chapitres I et II, Actual. Sci. Ind., no. 858, Hermann, Paris, 1940, pp. 52–59; these Rev. **3**, 55] that for each equivalence relation E over C there is a natural topology on the quotient space (C/E), namely the least topology on (C/E) which makes the natural projection of C onto (C/E) continuous, and that this topology on (C/E) is a Hausdorff topology if and only if E is a closed subset of $(C \times C)$.

The Lebesgue equivalence L is the set of all ordered pairs $(a, b) \in (C \times C)$ such that there is a homeomorphism h of X onto X with $a=bh$. It is well-known that L is not closed. It is shown that the closure of L is an equivalence relation, usually called Fréchet equivalence. It occurs to the reviewer that, though the closure of an equivalence relation is in general not an equivalence relation, the following statement holds. If G is a group of isometries of a metric space and if E is the equivalence relation usually called "congruence with respect to G," that is, $(a, b) \in E$ if and only if $a=g(b)$ for some $g \in G$, then the closure of E is an equivalence relation. This applies to our special case because to each

homeomorphism h of X onto X corresponds the isometry of C which maps a onto ah, and L is the congruence with respect to this group.

The Kerékjártó equivalence K is the class of all $(a, b)\varepsilon(C\times C)$ such that there exist a Peano space M, monotone maps m, n of X onto M, and a light map l of M into Y such that $a=lm$, $b=ln$. The author proves that K is a closed equivalence relation. *H. Federer.*

Mickle, Earl J. An extremal property of monotone-light factorizations. Duke Math. J. 16, 179–187 (1949).

This paper deals with the family $\mathcal{T}$ of all mappings from a Peano space X into a metric space Y. If $T\varepsilon\mathcal{T}$, an unrestricted factorization of T is a pair (f, s) consisting of a mapping $f: X\to\mathfrak{M}$, where $\mathfrak{M}$ is Peanian, and a second mapping $s: \mathfrak{M}\to Y$, such that $T=sf$. The factorization is monotone-light if f is monotone and onto, while s is light. The author considers a nonnegative function $\Phi(T)$, $T\varepsilon\mathcal{T}$ (infinite values permitted), having the following property: if $\mathfrak{C}$ is a true cyclic element of $\mathfrak{M}$ while $r_{\mathfrak{C}}: \mathfrak{M}\to\mathfrak{C}$ is the monotone retraction onto $\mathfrak{C}$ then $\psi(f, s)\equiv\sum\Phi(sr_{\mathfrak{C}}f)\geqq\Phi(T)$, the summation being taken over the class of true cyclic elements of $\mathfrak{M}$. The key result is the theorem that, if X is unicoherent, and (f, s) is an unrestricted factorization while (M, L) is a monotone-light factorization of $T\varepsilon\mathcal{T}$, then $\psi(f, s)\leqq\psi(M, L)$. This extremal property of monotone-light factorizations is used to show that a definition of enclosed volume employed by the reviewer for the spatial isoperimetric inequality is the largest in a wide class of possible definitions investigated jointly by Radó and the author.

J. W. T. Youngs (Bloomington, Ind.).

Theory of Functions of Complex Variables

Goodstein, R. L. A necessary and sufficient condition for differentiability. Edinburgh Math. Notes 37, 13–16 (1949).

Pour qu'une fonction complexe $f(z)$ soit différentiable dans un carré S, il suffit qu'à tout $\epsilon>0$ on puisse faire correspondre une subdivision de S en un nombre fini de carrés s_r, de côté δ_r, telle que pour chaque valeur de r il existe un nombre A_r et que $|f(z)-f(z')-A_r(z-z')|<\epsilon\delta_r$ pour tout couple z, z' contenu dans s_r. De ce lemme et de ses généralisations, l'auteur tire une condition suffisante pour que $f(z)$ soit constante dans S. *J. Lelong* (Caen).

Ahlfors, Lars, and Heins, Maurice. Questions of regularity connected with the Phragmén-Lindelöf principle. Ann. of Math. (2) 50, 341–346 (1949).

The principal results are as follows. Let $u(z)$ ($\not\equiv-\infty$) be subharmonic in $x>0$ and satisfy $\limsup u\leqq 0$ as $z\to iy+0$ for each real y. Let $\alpha=\sup u/x<+\infty$ for $x>0$. Then (A) for all θ ($|\theta|<\pi/2$) except for a set of outer logarithmic capacity zero, $\lim_{r\to\infty} u(re^{i\theta})/r=\alpha\cos\theta$; (B) this holds uniformly in $|\theta|<\theta_0<\pi/2$ if r is restricted to be outside a set of finite logarithmic length. Result (A) cannot be improved, and r in (B) cannot be replaced by a function which is $o(r)$. [Cf. J. Lelong, C. R. Acad. Sci. Paris 226, 1161–1163 (1948); these Rev. 10, 39.] *R. P. Boas, Jr.* (Providence, R. I.).

Garabedian, P. R., and Schiffer, M. Identities in the theory of conformal mapping. Trans. Amer. Math. Soc. 65, 187–238 (1949).

Let D be a bounded region in the complex z-plane bounded by n smooth closed curves C_ν ($\nu=1, 2, \cdots, n$). The relations between various domain functions of D are discussed. It is well known that there exist the following two univalent functions $\Phi(z; u, v)$ and $\Psi(z; u, v)$ in D: $\Phi(z; u, v)$ maps D upon the entire complex plane slit along concentric circular arcs around the origin so that the point u in D goes into the origin and the point v in D goes into the point at infinity; the residue of $\Phi(z; u, v)$ at the simple pole v is 1; $\Psi(z; u, v)$ maps D on the entire complex plane slit along rectilinear segments directed towards the origin so that u in D goes into the origin and v in D goes into the point at infinity; the residue of $\Psi(z; u, v)$ at the simple pole v is 1. The pair of functions $P(z; u, v)=\frac{1}{2}[\log\Phi(z; u, v)-\log\Psi(z; u, v)]$ and $Q(z; u, v)=\frac{1}{2}[\log\Phi(z; u, v)+\log\Psi(z; u, v)]$ is studied in detail. The relation $P(z; u, v)=-Q(z; u, v)+k_\nu(u, v)$ is established for z on C_ν, where $k_\nu(u, v)$ is independent of z. Let Ω_ν be the class of all functions $f(z)$ which are regular and single-valued in D, vanish at the point v of D, and for which the integral

$$(f, f)=\iint_D |f'(z)|^2\,dx\,dy, \qquad z=x+iy,$$

taken in the Lebesgue sense, exists and is finite. It is shown that $P(z; u, v)$ is the kernel function of every complete orthonormal system with respect to Ω_ν. Some extremal problems in conformal mapping are treated. Among several results, we mention the following. All functions $f(z)$ which are regular and single-valued in D and satisfy $f(u)=1$ and $f(v)=0$ map D on a region whose area is at least $\pi/P(u; u, v)$. The extremal map is obtained by means of the function $f_0(z)=P(z; u, v)/P(u; u, v)$. A representation $E(z; u, v)=P'(z; u, v)/Q'(z; u, v)$ is obtained for a function which maps D on the unit circle covered $2n$ times. Systematic use is made of contour integration and the theory of residues. The following symmetry laws for the functions $P(z; u, v)$, $Q(z; u, v)$ are obtained:

$$P(a; u, v)-P(b; u, v)=\overline{P(u; a, b)}-\overline{P(v; a, b)},$$
$$Q(a; u, v)-Q(b; u, v)=Q(u; a, b)-Q(v; a, b).$$

Analogous results are obtained for another pair of functions $M(z; u)$ and $N(z; u)$. These are again defined by means of two mapping functions, $A(z; u)$ which maps D on the complex plane slit along rectilinear segments parallel to the real axis and has a simple pole at u with residue 1, and $B(z; u)$ which maps D on the complex plane slit along rectilinear segments parallel to the imaginary axis and has a simple pole at u with residue 1. Then $M(z; u)$ and $N(z; u)$ are defined by setting $M(z; u)=\frac{1}{2}[A(z; u)-B(z; u)]$ and $N(z; u)=\frac{1}{2}[A(z; u)+B(z; u)]$. Various extremal problems are again considered, whose solutions are given in terms of the functions M and N. The connection between the two pairs P, Q and M, N is given by the relations:

$$\overline{P'(a; u, v)}=M(u; a)-M(v; a),$$
$$Q'(a; u, v)=N(v; a)-N(u; a).$$

Now let $\omega_\nu(z)$ be harmonic in D and have on C_μ the boundary values $\delta_{\mu\nu}$ ($\delta_{\mu\nu}$ being the Kronecker symbol), and let $w_\nu(z)$ be defined up to an imaginary additive constant by $\omega_\nu(z)=\Re\{w_\nu(z)\}$. If z describes a circuit around the contour C_μ, $w_\nu(z)$ increases by the period $-2\pi i P_{\mu\nu}$. It is shown that $w_\nu(u)-w_\nu(v)=\sum_{\mu=1}^{n} k_\mu(u, v)P_{\mu\nu}$. The Green's function $g(z; \zeta)$ and Neumann's function $\gamma(z; \zeta)$ of D are completed by means of the analytic functions $p(z; \zeta)$ and $\pi(z; \zeta)$ defined, to within additive imaginary constants, by $g(z; \zeta)=\Re\{p(z; \zeta)\}$ and $\gamma(z; \zeta)=\Re\{\pi(z; \zeta)\}$. Relations between the functions P, Q, p, π, w_ν are obtained. The connec-

tion of these domain functions with Schottky functions [J. Reine Angew. Math 83, 300–351 (1877)] and other related classes of functions is discussed. The analogy of the domain functions with the theory of Abelian integrals is stressed throughout. Finally, the theory of variation of the domain functions under an infinitesimal change of the domain D, previously introduced by one of the authors [M. Schiffer, Amer. J. Math. 68, 417–448 (1946); these Rev. 8, 325], is discussed. *W. Seidel.*

Schwartz, Marie-Hélène. Sur les surfaces de Riemann possédant des points critiques arbitrairement rapprochés. C. R. Acad. Sci. Paris 228, 154–155 (1949).

Two examples of parabolic covering surfaces of the sphere with irregular behavior. First, a surface for which a certain defect which generalizes a similar notion of Ahlfors is greater than its normal value 2. Second, a surface corresponding to a function $f(z)$ with noncompact regions $|f(z)-A|\leqq\sigma(|z|)$, $\sigma(r)\to 0$ for $r\to\infty$, such that A is neither an asymptotic value nor a limit of asymptotic values. The mutual distances of the critical points of such surfaces cannot be bounded from below. The second example answers a query of Collingwood [same C. R. 227, 813–815 (1948); these Rev. 10, 363]. *L. Ahlfors* (Cambridge, Mass.).

Fourès, Léonce. Décomposition en feuillets des surfaces de Riemann de type parabolique. C. R. Acad. Sci. Paris 228, 644–646 (1949).

By use of the theorem of Gross a system of fundamental regions of an entire function is constructed with the property that all fundamental regions have star-shaped images with the same center. It is pointed out that this subdivision is sometimes improper. *L. Ahlfors* (Cambridge, Mass.).

Fourès, Léonce. Sur les points transcendants de la fonction inverse d'une fonction entière $w=f(z)$. C. R. Acad. Sci. Paris 228, 734–736 (1949).

Construction of a system of fundamental regions for an entire function whose inverse function has only isolated transcendental singularities. *L. Ahlfors.*

Thiem, Le Van. Le degré de ramification d'une surface de Riemann et la croissance de la caractéristique de la fonction uniformisante. C. R. Acad. Sci. Paris 228, 1192–1195 (1949).

It is well known that a high degree of ramification of a simply connected Riemann surface favors the hyperbolic case, while a slow ramification leads to the parabolic type. The author makes the remark that an intensification of the ramification accelerates the growth of Nevanlinna's characteristic function $T(r)$ in the parabolic case, but retards it in the hyperbolic case.The remark is illustrated by examples. *L. Ahlfors* (Cambridge, Mass.).

Thiem, Le Van. Un problème de type généralisé. C. R. Acad. Sci. Paris 228, 1270–1272 (1949).

The author determines the generalized type (zero or positive boundary in the sense of Nevanlinna) of two classes of Riemann surfaces of infinite genus for which Nevanlinna's criterion [C. R. Dixième Congrès Math. Scandinaves 1946, pp. 225–240 (1947); these Rev. 8, 509] fails. One class is represented by a regular graph of order three and proved to be of positive type by a theorem of Myrberg [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 10 (1941); these Rev. 7, 516]. The second is represented by combs of higher but finite order: a comb of higher order is one whose teeth are again combs, and so forth. These surfaces are of zero type. *L. Ahlfors* (Cambridge, Mass.).

Radojčić, M. Sur un problème topologique de la théorie des surfaces de Riemann. Acad. Serbe Sci. Publ. Inst. Math. 2, 11–25 (1948). (French. Serbian summary)

The author considers sheets of a simply connected Riemann surface or, equivalently, fundamental regions in the domain on which it is mapped. A rather detailed study of the structure of the net of fundamental regions is undertaken which parallels the corresponding treatment of automorphic functions. The fundamental regions have vertices which correspond to algebraic branch-points and transcendental singularities on the surface, and certain limitations concerning the location of such vertices are derived. The results are topological in nature. *L. Ahlfors.*

Vostrecov, B. A. On the existence of boundary values and on the integral representation of functions analytic in the unit circle. Doklady Akad. Nauk SSSR (N.S.) 65, 7–8 (1949). (Russian)

(1) If $\sum a_n z^n$ converges in $|z|<1$, then it is possible to find constants $A_n>|a_n|$ and such that $\sum A_n z^n$ is regular in $|z|<1$ and has boundary values at almost all points of the unit circle. (2) If $f(z)$ is regular in $|z|<1$, then

$$f(z)=(2\pi i)^{-1}\int_{|\varphi|=1}\omega(\varphi)\varphi^{-1}g(\varphi/(\varphi-z))d\varphi,$$

where $\omega(\varphi)=\omega(e^{i\theta})\in L^2(0,2\pi)$ and $g(t)$ is an integral function of t. The proofs are based on Wigert's theorem on $\sum\psi(n)z^n$ ($\psi(z)$ an integral function of order one, minimal type) and the remark that for any sequence $\{a_n\}$ with $\limsup|a_n|^{1/n}\leqq 1$ there is an integral function $\psi(z)$ of order one and minimal type with $\psi(n)>|a_n|$. *W. H. J. Fuchs* (Liverpool).

Germansky, Baruch. On the systems of Fekete-points of an arc of a circumference. Riveon Lematematika 3, 1–7 (1949). (Hebrew)

Detailed proofs of results announced in C. R. Acad. Sci. Paris 206, 1163–1166 (1938). *Z. Nehari.*

Schaeffer, A. C., and Spencer, D. C. The coefficients of schlicht functions. IV. Proc. Nat. Acad. Sci. U. S. A. 35, 143–150 (1949).

[For previous parts cf. Duke Math. J. 10, 611–635 (1943); 12, 107–125 (1945); same Proc. 32, 111–116 (1946); these Rev. 5, 175; 6, 206; 7, 424.] Let S denote the class of functions $w=f(z)=\sum_1^\infty a_n z^n$, $a_1=1$, regular and schlicht in $|z|<1$, and let V_n be the closed region in E_{2n-2} of all points $(a_2, a_3, \cdots, a_n)$ belonging to $f(z)$ of S. In part III it was pointed out that any function $f(z)$ corresponding to a boundary point of V_n satisfies a differential equation of the form

$$\left(\frac{z}{w}\,\frac{dw}{dz}\right)^2 P(w)=Q(z),$$

where $P(w)$, $Q(z)$ are certain rational functions in w and z, respectively. Functions $f(z)$ of class S which satisfy this differential equation are called D_n functions. In part IV the authors show that there is a function

$$v(z,u)=u(z+\sum_2^\infty b_n(u)z^n), \qquad 0<u\leqq 1,$$

regular and schlicht in $|z|<1$, which maps $|z|<1$ onto

$|v|<1$ minus a finite set of analytic arcs, and satisfies

$$\left(\frac{z}{v}\frac{\partial v}{\partial z}\right)^2 Q(v,u)=Q(z),$$

where $Q(v,u)=\sum_{-n-1}^{n-1}B_s(u)v^s$, $B_{-s}(u)=\overline{B_s(u)}$, and $Q(v,u)\geqq 0$ on $|v|=1$ with at least one zero there. The function $v(z,u)$ is continuous with respect to u, differentiable except at a finite number of points, and satisfies Löwner's differential equation:

$$\frac{u}{v}\frac{\partial v}{\partial u}=\frac{1+ve^{i\omega(u)}}{1-ve^{i\omega(u)}},$$

with $Q(e^{-i\omega(u)},u)=0$, and where

$$v(z,1)=z,\quad \lim_{u\to 0}v(z,u)/u=f(z),\quad Q(z,1)=Q(z),$$

$$\lim_{u\to 0}Q(uw,u)=P(w).$$

As u varies from 1 to 0, the point $(b_2(u),\cdots,b_n(u))$ traces a curve from the origin to a boundary point of V_n. For $n=4$, these curves have been used in numerical calculations to determine the boundary of V_4 in an attempt to prove $|a_4|\leqq 4$. The computational program underway consists of three parts: (a) proof that points of the boundary of V_4 corresponding to $z(1-e^{i\theta}z)^{-2}$ have the local maximum property with respect to neighboring points; (b) proof that over a large part of the boundary $|a_4|\leqq 4-\delta$, $\delta>0$; (c) calculation of points on the remainder of the boundary by integrating a system of differential equations. Some results are obtained concerning the boundary of the general region V_n, but space does not permit a precise statement here in view of the technicalities involved. *M. S. Robertson.*

***Hamdi Alisbah, Orhan. Über starkschlichte Abbildung des Einheitskreises.** Université d'Istanbul. Faculté des Sciences. Recueil de mémoires commémorant la pose de la première pierre des Nouveaux Instituts de la Faculté des Sciences, pp. 39–44, Istanbul, 1948.

When $f(z)=\sum_1^\infty a_nz^n$ is regular for $|z|<1$ and $D(z,\xi)\neq 0$ for all $|z|<1$, $|\xi|\leqq 1$, where $D(z,\xi)=(f(z)-f(\xi z))/(z-\xi z)$, $f(z)$ is then schlicht for $|z|<1$. If further $|D(z,\xi)|\geqq m>0$ for all $|z|<1$, $|\xi|\leqq 1$, then $f(z)$ is strongly schlicht for $|z|<1$. The author considers the class of functions $f(z)$ normalized so that $m=1$. Thus $|D(z,\xi)|^{-1}\leqq 1$ for $|z|<1$. It is then shown that (1) $|a_2|\leqq|1+\xi|^{-1}(|a_1|^2-1)$ for all $|\xi|\leqq 1$. The reviewer notes that (1) is an immediate consequence of Schur's inequalities for the coefficients of bounded analytic functions [J. Reine Angew. Math. **147**, 205–232 (1917); **148**, 122–145 (1918)]. If $f(z)$ is schlicht for $|z|<1$ and omits the value c when $|z|<1$ then (2) $|c|\geqq|a_1|^2/(2|a_1|+|a_2|)$ ($\geqq\frac{1}{4}$ for $|a_1|\geqq 1$). For the normalized strongly schlicht functions (1) (with $\xi=1$) and (2) give $|c|\geqq 2|a_1|^2/(4|a_1|+|a_1|^2-1)\geqq\frac{1}{3}$. *M. S. Robertson* (New Brunswick, N. J.).

Cowling, V. F. Analytic continuation of factorial series. Amer. J. Math. **71**, 283–286 (1949).

The author continues his study of series whose coefficients are the values of an analytic function. Let $h>0$ and $0<\beta\leqq\pi$. Let $a(w)$ be analytic in the sector $\arg(w-h)\leqq\beta$, and there satisfy the condition $a(h+re^{i\psi})=O(r^k\exp(-Lr\sin\psi))$ for some L, $0<L<2\pi$, and k, and all large r. Let

$$f(z)=\sum_0^\infty a(n)n!/z(z+1)(z+2)\cdots(z+n),$$

with finite abscissa of convergence. Then f can be extended to the whole plane, except possibly for the points $z=0,-1,\cdots$. *R. C. Buck* (Providence, R. I.).

Walsh, J. L., and Nilson, E. N. On functions analytic in a region: approximation in the sense of least pth powers. Trans. Amer. Math. Soc. **65**, 239–258 (1949).

Let C_1 and C_0 be finite sums of disjoint Jordan curves, and let the finite region R be bounded by C_1, S by C_0, where S is interior to R and closed; let $f(z)$ be analytic on S, but not throughout R. In previous papers [e.g., Nilson, Trans. Amer. Math. Soc. **55**, 53–67 (1944); these Rev. **5**, 115] the authors have dealt with functions $f_M(z)$ of best approximation to $f(z)$ on S which were required to be analytic and of modulus not greater than M ($M>0$ given) on R. Here the L_p theory of approximation on S is given; existence, uniqueness and continuity properties of the f_M are discussed. Introducing the functions $\phi(z)$ and $\psi(z)$, where ϕ is harmonic in $R-S$, continuous on $\overline{R-S}$, 0 on C_0, 1 on C_1, ψ conjugate to ϕ, they consider the class (K_p, say) of functions $F(z)$, analytic in R and with uniformly bounded means

$$\mu_p(F,\nu)=\left\{\tau^{-1}\int_{C_\nu}|F(z)|^p d\psi(z)\right\}^{1/p},$$

$0<\nu<1$, $p>0$ fixed; C_ν is the curve $\Phi(z)=\nu$, τ is the total variation of ψ along C_1. Starting thus they generalise the Riesz theory of the class H_p for the region R, and the Hardy mean-value theorem for $R-S$. Given positive numbers M, p, q, there exists a solution $\mathfrak{F}_M(z)$ of the minimum problem: $F\varepsilon K_q$, $\mu_q(F,1)\leqq M$, $\mu_p(f-F,0)=\text{minimum}=m_M$ = measure of the "best approximation" to $f(z)$. Then $\mathfrak{F}_M$ is unique when $1\leqq p<\infty$, $1\leqq q\leqq\infty$. Under some conditions on f, $\mathfrak{F}_M\to f$ ($M\to\infty$) uniformly on S,

$$\limsup\{\mu_t(f-\mathfrak{F}_M,\sigma)\}^{1/\log M}=e^{(\sigma-\rho)/(1-\rho)}$$

($M\to\infty$) for each σ,t ($0\leqq\sigma<\rho$, $0<t\leqq\infty$; ρ is some constant, $0<\rho<1$). In addition, a related problem is dealt with: $F\varepsilon K_q$, $\mu_p(f-F,0)\leqq m$ ($m>0$ given), $\mu_q(F,1)=\text{minimum}$. If $p=q=2$ and C_0, C_1 are the circles $|z|=r_0$, r_1 ($0<r_0<1<r_1$), $\mathfrak{F}_M(z)$ is obtained explicitly, and the precise measure of approximation is found, including some identities. A necessary and sufficient condition that $f(z)$ be of lacunary structure is that $\liminf m_M^{1/\log M}<\limsup m_M^{1/\log M}$ ($M\to\infty$). Finally m_M is shown to be a continuous functional of $f(z)$ and M. *H. Kober* (Birmingham).

Pastidès, Nicolas. Sur quelques équations fonctionnelles. Ann. Sci. École Norm. Sup. (3) **65**, 277–298 (1948).

On pose $f_2(z)=f(f(z))$, $f_n(z)=f(f_{n-1}(z))$, $f_{-1}(z)$ est l'inverse de $f(z)$. L'auteur montre que: (I) si $f(z_0)=z_0$ et si D est un domaine renfermant z_0 dans lequel $f(z)$ est holomorphe, si $f_n(z)\equiv z$, il existe un domaine simplement connexe E contenant z_0 qui est transformé biunivoquement en lui-même par la substitution $(z,f(z))$. Par représentation conforme de E sur un cercle, il s'ensuit que, dans les conditions de (I), on a (II): il existe une infinité de fonctions $F(z)$ holomorphes et univalentes autour de z_0 telles que

$$f(z)\equiv F_{-1}[sF(z)],\quad F(z_0)=0,\quad f'(z_0)=s,$$

s étant une racine d'ordre n de l'unité, autre que 1. L'auteur détermine alors toutes les fonctions $F(z)$ jouissant de cette propriété; si $G(z)$ est l'une d'elles, toutes les autres sont de la forme $\sum c_qG(z)^{nq+1}$, la série $\sum c_qu^{nq+1}$ ayant un rayon de convergence non nul. Ce résultat repose sur l'étude des solutions des équations $F(f(z))\equiv sF(z)$ où $f(z)$ est la fonction ci-dessus et $F(z)$ la fonction inconnue, et de leur

transformées $f(\varphi(z))\equiv\varphi(sz)$, $s=f'(0)$, $z_0=0$, où φ est la fonction inconnue. Ces résultats étendent à ce cas où s a la valeur indiquée, les théorèmes de Koenigs relatifs au cas $|s|\neq 1$ [voir Fatou, Bull Soc. Math. France **47**, 161–271 (1919); **48**, 33–94, 208–314 (1920)]. L'auteur étudie aussi les équations fonctionnelles $\psi(f(z))\equiv g(\psi(z))$ lorsque g et f données sont holomorphes et nulles à l'origine avec des dérivées égales à s en ce point, $|s|\neq 1$, soit dans le cas de Koenigs, $s\neq 0$, soit dans le cas de Boettcher $s=0$.

G. Valiron (Paris).

Ghermanescu, Michel. Opérateurs fonctionnels périodiques. C. R. Acad. Sci. Paris **228**, 1190–1192 (1949).

L'auteur considère une fonction méromorphe dans tout le plan à distance finie et la suite de ses itérées, $\theta_1\equiv\theta(z)$, $\theta_2\equiv\theta(\theta_1(z))$, $\theta_n\equiv\theta(\theta_{n-1}(z))$; il montre que l'identité (1) $\theta_n(z)\equiv z$ ne peut avoir lieu que si $\theta(z)$ est de la forme $(az+b)/(cz+d)$, a, b, c, d étant des constantes vérifiant certaines conditions. Il traite d'abord le cas où $\theta(z)$ est une fraction rationnelle. Il observe que, au cas où (1) a lieu avec $n=2$ ou $n=3$, correspond des équations intégrales résolubles par une méthode de Parodi. [Signalons que le cas où $\theta(z)$ est fraction rationnelle avait été liquidé par Julia [voir J. Math. Pures Appl. (7) **4**, 47–245 (1918), voir p. 82] et que l'impossibilité de (1) lorsque $\theta(z)$ est une fonction entière résulte des considérations de Fatou [Acta Math. **47**, 337–370 (1926)].]

G. Valiron (Paris).

Sunyer i Balaguer, Ferran. Une généralisation des fonctions presque-périodiques. C. R. Acad. Sci. Paris **228**, 732–734 (1949).

Using the spherical distance between two complex numbers z_1 and z_2 instead of $|z_1-z_2|$ the author generalizes the notion of almost periodic function in a strip from holomorphic to meromorphic functions. The fact, however, that the sum and the product of two such functions (belonging to the same strip) need not be a function of the same sort, seems to lessen the interest of these functions. The results of Favard [same C. R. **194**, 1714–1716 (1932)] and some of the results of Jessen [Math. Ann. **108**, 485–516 (1933)] are generalized. *E. Følner* (Copenhagen).

Sunyer i Balaguer, Ferran. Une généralisation des fonctions presque-périodiques: fonctions presque-elliptiques. C. R. Acad. Sci. Paris **228**, 797–799 (1949).

Assuming the strip in the paper reviewed above to be the whole plane and almost periodicity to hold in two directions, the author obtains a certain generalization of elliptic functions. The author was not aware that these "almost elliptic" functions were treated previously by Bessonof [same C. R. **182**, 1011–1013 (1926); **186**, 63–65 (1928)], Favard [ibid. **185**, 1434–1436 (1927)], and Norgil [Mat. Tidsskr. B. **1930**, 73–91]. One of the results is an answer to what becomes of the first, second, and third Liouville theorems in case of these functions. *E. Følner.*

Nevanlinna, Rolf. Über die Neumannsche Methode zur Konstruktion von Abelschen Integralen. Comment. Math. Helv. **22**, 302–316 (1949).

The author treats by means of integral equations the Neumann method for the construction of Abelian integrals. The problem is reduced with the aid of harmonic measure to the study of

$$u(x)=u_0(x)+\int_0^1 u(y)d\varphi(y,x),$$

where φ is given in $S:[0\leqq x,y\leqq 1]$, with φ and φ_y both continuous and $\varphi_y\geqq 0$ in S, $\varphi(1,x)-\varphi(0,x)=1$ for $0\leqq x\leqq 1$; and where u_0 is given integrable on $[0,1]$. The solution of this equation is treated in detail, and the results of the study are applied to Neumann's problem. Application is made to certain classes of automorphic functions. *M. Heins.*

Pic, Georges. Sur une équation fondamentale relative aux groupes finis de substitutions linéaires. C. R. Acad. Sci. Paris **228**, 1268–1270 (1949).

Let S denote a linear substitution of the type

$$z_i'=\frac{a_{i1}z_1+a_{i2}z_2+\cdots+a_{in}}{a_{n1}z_1+a_{n2}z_2+\cdots+a_{nn}},\quad i=1,2,\cdots,n-1.$$

The author considers finite groups G of substitutions S, and investigates, in particular, the fixed points of such substitutions, that is, points P such that $P=S(P)$. Two fixed points P and Q are called equivalent if there exists a substitution T of the group G such that $Q=T(P)$. All substitutions of G which have a common fixed point (equivalent points being considered identical) form a subgroup G' which is homomorphic to the group of linear substitutions of $n-2$ variables. The main result of the paper is a formula connecting the order of G with the orders of the subgroups G' belonging to all the fixed points of all substitutions of G. This generalizes a formula established by F. Klein for the case $n=2$.

Z. Nehari (St. Louis, Mo.).

Myrberg, P. J. Über gewisse Cremonagruppen und ihre automorphen Funktionen. II. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **53**, 14 pp. (1948).

[For part I cf. the same Ann. Ser. A. **23** (1924), no. 9 (1925).] The theory of automorphic functions is concerned with analytic functions which are invariant with respect to a given group of biuniform analytic transformations. While in the case of one complex variable these transformations are necessarily linear, there exist in the case of two or more variables nonlinear birational transformations. The investigation of groups of such transformations (Cremona groups) and their relation to the theory of automorphic functions has so far almost exclusively been confined to the case in which the transformations of the group are rational functions of bounded degree. In the present paper, the author discusses a group of Cremona transformations which does not fall within this category and which is closely related to the modular group. This group consists of transformations of the form $x'=x^\alpha y^\beta$, $y'=x^\gamma y^\delta$, where the exponents are rational numbers satisfying the condition $\alpha\delta-\beta\gamma=1$. The discussion of this group is followed by the construction of automorphic functions invariant with respect to it.

Z. Nehari (St. Louis, Mo.).

Petersson, Hans. Automorphe Formen als metrische Invarianten. II. Multiplikative Differentiale als Grenzwerte metrischer Invarianten von stetig veränderlicher reeller Dimension. Math. Nachr. **1**, 218–257 (1948).

[For part I, see the same vol., 158–212 (1948); these Rev. **10**, 365.] Let Γ be a properly discontinuous group of matrices

$$L=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

with real elements α, β, γ, δ and determinant 1, which contains parabolic substitutions whose fixed points are everywhere dense on the real axis. The entire Poincaré series

referred to in the title are defined as follows:

$$(1)\qquad G_{-r}(\tau, v_r, A, \Gamma, m)=\sum_{M}\frac{e^{2\pi i M\tau(s+m)/N}}{v_r(M)(m_1\tau+m_2)^r},$$

where $|v_r|=|v_r(M)|=1$; A is a real unimodular 2-rowed matrix such that $A^{-1}(\infty)$ is a parabolic vertex of Γ; m is a nonnegative integer; κ is defined by $0\leqq\kappa<1$; $v(P)=\exp(2\pi i\kappa)$, where $P=A^{-1}U^NA$, $U^N\tau=\tau+N$, and N is the smallest positive integer for which $P\subset\Gamma$; and

$$M=\begin{pmatrix}\cdots & \cdots\\ m_1 & m_2\end{pmatrix}$$

runs over a complete system of matrices in $A\Gamma$ with different second line. Then G_{-r} is an entire automorphic form of negative dimension $-r$; it is regular in $\Im(\tau)>0$, and satisfies

$$(2)\qquad G_{-r}(L\tau)=v_r(L)(\gamma\tau+\delta)^rG_{-r}(\tau)$$

for every $L\subset\Gamma$. When $r=2$, the above series no longer converges uniformly or absolutely and so cannot be used to represent forms of dimension -2; the latter are particularly interesting since they are the multiplicative (in special cases, the Abelian) differentials on the Riemann surface defined by the fundamental region of Γ. Starting with a given group Γ and a given "multiplier" $v_2=v_2(L)$, one defines a family of multipliers $v_r(L)$, depending continuously on the real dimension $-r$, so that $v_r(L)\to v_2(L)$ as $r\to 2+0$, for every $L\subset\Gamma$. The corresponding Poincaré series G_{-r} formed with these multipliers v_r is also a continuous function of r for $r>2$. The author sets himself the task of discovering whether $\lim_{r\to 2+0}G_{-r}$ exists, and if so, what its properties are. A direct limit process $r\to 2$ is not available because of the convergence failure of the series for G_{-2}.

The principal result is as follows: if the limiting process is suitably restricted, then $\lim_{r\to 2+0}G_{-r}(\tau, v_r, A_j, \Gamma, 0)$ exists and is an entire automorphic form of dimension -2 with the preassigned multiplier v_2. Furthermore, the limit function is orthogonal to every entire cusp-form of dimension -2 having the given multiplier v_2. (A cusp-form is one which vanishes at all parabolic vertices of Γ; orthogonality of forms will be explained below.)

Sketch of one case of proof. In previous papers the author has constructed a metric space out of the family $\mathfrak{E}(-r, v_r)$ of entire forms in $\{\Gamma, -r, v_r\}$ (these are the functions of τ, regular in the upper half-plane, which satisfy (2)), and has defined a scalar product of forms in $\mathfrak{E}$. Let the preassigned v_2 satisfy $v_2(L)\neq 1$. The first step of the proof consists in finding a continuous finite basis for the family $\mathfrak{E}(-r, v_r)$. The essential idea, which follows easily from theorems 2a and 9 of part I, is that the Poincaré series $G_{-2-\rho}(\tau, v_r, A_h, \Gamma, 0)$ and $G_{-2-\rho}(\tau, w_r, A_0, \Gamma, m_i)$, where h and i run over certain finite sets of integers and ρ is fixed and positive, form a basis for the family of forms $\mathfrak{E}(-2-\rho, w_r)$ in the range $2\leqq r\leqq r_0$ for a certain r_0; w_r is explained below. (Note that the dimension $-2-\rho$ of G is now divorced from the parameter r of w_r.) The G series are continuous functions of τ and r. By multiplying G by a form (continuous in r) of dimension $2+\rho-r$ and suitable multiplier, one arrives at a continuous basis of the entire forms of dimension $-r$ and preassigned multiplier v_r. (The forms needed above as multipliers of G are obtained from the theory of differentials on the closed Riemann surface defined by the fundamental region of Γ; w_r is chosen so that, after multiplication by these forms, one arrives at the multiplier v_r.) The finite set of normed basis forms are denoted by $\chi_h(\tau, r)$ and $\psi_k(\tau, r)$. The author proves that all scalar products formed from χ_h and ψ_k are continuous functions of r in $2\leqq r\leqq r_0$. Thus we have, for a fixed j, the representation as a finite sum

$$(3)\qquad G_{-r}(\tau, v_r, A_j, \Gamma, 0)=\sum_h\xi_{j,h}(r)\chi_h(\tau, r)+\sum_k\eta_{j,k}(r)\psi_k(\tau, r)$$

in $2<r\leqq r_0$, but the second term vanishes by virtue of the orthogonality properties of G_{-r}, χ_h, and ψ_k. By scalar multiplication of the resulting equation by $\chi_i(\tau, r)$, $i=1, 2, \cdots, s$, one obtains a system of linear equations for $\xi_{j,h}$, which the author then solves by evaluating the determinant of the scalar products. This results in an explicit form for $\xi_{j,h}(r)$ from which it is seen that it is continuous at $r=2+0$. The limit $r\to 2+0$ can then be taken in (3) term by term and furnishes the quoted result. *J. Lehner.*

Igusa, Jun-ichi. Zur klassischen Theorie der algebraischen Funktionen. J. Math. Soc. Japan 1, 63–72 (1948).

The author presents a brief outline of the theory of algebraic functions of one variable over the complex number field and indicates some simplifications over the classic presentation of the theory. Abel's theorem and the inversion theorem of Jacobi are interpreted from the viewpoint of the character theory of Abelian groups. Thus the suitably topologized group of divisors of degree zero and the (discrete) Betti group are shown to form an orthogonal pair. A further use of the unramified multiplicative functions sets up a similar pairing between the Galois group of the maximal unramified Abelian extension and the group of divisors of degree zero and finite order.

O. F. G. Schilling (Chicago, Ill.).

Položil, G. N. On the application of the generalized derivative to a class of quasi-conformal mappings. Doklady Akad. Nauk SSSR (N.S.) 63, 615–618 (1948). (Russian)

If (1) $u=u(x, y)$, $v=v(x, y)$ are sufficiently smooth functions, then at each point $z_0=x_0+iy_0$ where $u_xv_y-u_yv_x>0$ the characteristics $p(z_0)$ and $\theta(z_0)$ of the mapping (1) are defined as the axes-ratio and slope of major axis, respectively, of the infinitesimal ellipse with center at z_0 which is mapped by (1) onto an infinitesimal circle. Lavrentyeff [Rec. Math. [Mat. Sbornik] (1) 42, 407–424 (1935)] proved that Riemann's mapping theorem holds for mappings with prescribed characteristics, provided $p(z)$ is uniformly continuous in the domain to be mapped and $\theta(z)$ is continuous for $p(z)\neq 1$. In the present note the author assumes that $p(x, y)$ and $\theta(x, y)$ are analytic functions of x and y, $x^2+y^2<1$. Set $p(x, y)=p^*(x+iy, x-iy)$, and set $\theta(x, y)=\theta^*(x+iy, x-iy)$. Denote by $\zeta=\Psi(z)$ a solution of the differential equation

$$\frac{d\zeta}{dz}=\frac{p^*(z, \zeta)+1}{p^*(z, \zeta)-1}e^{-2i\theta^*(z, \zeta)}.$$

Then $\gamma=\alpha+i\beta=\bar{z}-\Psi(z)$ is a mapping of $|z|<1$ onto some domain G in the γ-plane with characteristics p and θ. If a mapping onto a given domain D is desired, it is merely necessary to map G conformally onto D. [The reduction of Lavrentyeff's mapping problem (for analytic p and θ) to a differential equation in the complex domain is not surprising since the problem is equivalent to the determination of isothermal coordinates on a two-dimensional Riemannian manifold.] The generalized derivative mentioned in the title is the formal derivative

$$\partial(u+iv)/\partial z=[\partial(u+iv)/\partial x]-i[\partial(u+iv)/\partial y].$$

L. Bers (Princeton, N. J.).

Lopatinskiĭ, Ya. B. On a generalization of the concept of analytic function. Doklady Akad. Nauk SSSR (N.S.) **64**, 155–158 (1949). (Russian)

Let A and B be linear differential operators,

$$A=\sum a_{kl}\partial^{k+l}/\partial x^k\partial y^l,\quad B=\sum b_{kl}\partial^{k+l}/\partial x^k\partial y^l,$$

$k+l\leqq n$, where a_{kl}, b_{kl} are real analytic functions of $(x, y)\epsilon\Omega$. It is assumed that the resultant of the forms $\sum a_{kl}\lambda^k\mu^l$, $\sum b_{kl}\lambda^k\mu^l$ does not vanish and that $AB=BA$. A function $w=u+iv$ analytic in $(x, y)\epsilon\Omega_1\subset\Omega$ is called (A, B)-analytic if (1) $Au=Bv$, $Bu=-Av$, which is equivalent to (2) $(A+iB)w=0$. If w is (A, B)-analytic, so is Aw, and for two (A, B)-analytic functions w and w_1 the ratio Aw/Aw_1 is, in general, the limit of a certain complicated generalized difference quotient. A point (x_0, y_0) is "ordinary" if a certain determinant of partial derivatives of A and B does not vanish at this point. In the neighborhood of an ordinary point an (A, B)-analytic function w admits a "Taylor expansion" in a series of certain simple (A, B)-analytic functions (independent of w) with coefficients determined by the values of the partial derivatives of w at (x_0, y_0). Using the concept of adjoint operators and of the fundamental solution of a partial differential equation the author also obtains a generalization of Cauchy's theorem and Cauchy's formula. No proofs are given.

Example. If $A=(\partial^2/\partial x^2)-(\partial^2/\partial y^2)$, $B=2\partial^2/\partial x\partial y$, then $(A+iB)w=0$ implies that $\Delta\Delta u=\Delta\Delta v=0$ and "Taylor's expansion" yields the Goursat representation of biharmonic functions. [While this paper has a formal similarity to the theory of Σ-monogenic functions [Bers and Gelbart, Trans. Amer. Math. Soc. **56**, 67–93 (1943); these Rev. **6**, 86], it should be noted that a Σ-monogenic function is analytic in the sense of the author only in the trivial case when the defining system of equations is reducible to the Cauchy-Riemann system.]

L. Bers (Princeton, N. J.).

Métral, Paul. Fonctions presque automorphes sur un anneau. C. R. Acad. Sci. Paris **228**, 800–801 (1949).

A complex-valued function $f(x)$ defined on a ring R is called a.p.l. to the right if the set of functions $\{f(xa+b)\,|\,a\epsilon R, b\epsilon R\}$ is totally bounded. The author states that the Schmidt-Peter-Weyl-Neumann method can be applied to these functions and yields the result that they can be uniformly approximated by linear combinations of coefficients in unitary representations of the ring. The reviewer remarks that this is certainly false, since a ring has no unitary representations at all (while for instance $f(x)=0$ is a.p.l. in every ring).

E. Følner (Copenhagen).

Métral, Paul. Fonctions p.p.l. dans un espace linéaire. C. R. Acad. Sci. Paris **228**, 892–894 (1949).

The author states that results from Bochner and von Neumann's theory of almost periodic functions on a group have analogues for certain functions on a ring which were termed "a.p.l." in the note reviewed above.

E. Følner (Copenhagen).

Pierucci, Mariano. Un tentativo di estensione del concetto di numero complesso e sue eventuali applicazioni alla fisica teorica. Atti Sem. Mat. Fis. Univ. Modena **1**, 5–11 (1947).

The author makes some general remarks about hypercomplex numbers of the form $\sum_{k=0}^{n}a_k i_k$, where the i_k are the nth roots of unity, and then lets $n\to\infty$ in a manner which is not made clear.

R. P. Boas, Jr. (Providence, R. I.).

Nicolesco, Miron. Sur une classe de fonctions non monogènes de deux variables complexes. C. R. Acad. Sci. Paris **228**, 1323–1325 (1949).

Consider the function $f(z, z')\equiv u(x, y, x', y')+iv(x, y, x', y')$ of the complex variables $z=x+iy$ and $z'=x'+iy'$, defined in a domain $\mathfrak{D}$, and consider the differential operators $L\equiv\partial^2/\partial x\partial x'+\partial^2/\partial y\partial y'$ and $L'\equiv\partial^2/\partial x'\partial y-\partial^2/\partial x\partial y'$. Then the function $(L+iL')f(z, z')$, where it exists, is called the areolar derivative of $f(z, z')$. Further, $f(z, z')$ is said to be pseudo-monogenic if and only if its areolar derivative vanishes throughout $\mathfrak{D}$. If $f(z, z')$ is pseudo-monogenic in $\mathfrak{D}$, then the function $\frac{1}{2}(Lf-iL'f)\equiv Df$ is called the derivative of $f(z, z')$. With these definitions in hand, the author proceeds to develop a theory of pseudo-monogenic functions, a theory that has a great deal in common with the classical theory of (ordinary) monogenic functions. For example, if $f(z, z')$ is pseudo-monogenic in $\mathfrak{D}$, then Df is pseudo-monogenic in $\mathfrak{D}$; if F denotes a family of functions pseudo-monogenic in $\mathfrak{D}$, and if F is uniformly bounded in $\mathfrak{D}$, then the family F is a normal family in the sense of Montel. Moreover, the real and imaginary coefficients $u(x, y, x', y')$ and $v(x, y, x', y')$ of a pseudo-monogenic function have properties completely analogous to those of ordinary harmonic functions. The author states a series of results; the proofs are to appear elsewhere.

M. Reade (Ann Arbor, Mich.).

Nef, Walter. Homogene Räume mit invarianter Metrik. Comment. Math. Helv. **22**, 215–231 (1949).

The paper is intended as a generalization of some of S. Bergman's work from analytic functions to functions on certain homogeneous spaces. The space Ω is locally Euclidean and has a simply transitive group T of transformations. Moreover, T shall have a discrete normal subgroup whose factor group is compact. It is proved, first, that the linear space of differentiable functions on Ω contains an invariant, final dimensional subspace, such that, for any point and any direction, at least one function has a nonzero derivative. In this subspace Bergman's minimum problem with a fixed partial derivative has a solution and yields an invariant Riemannian metric. The author now wishes to generalize the Pick-Schwarz lemma to this metric. He succeeds under a restrictive condition which is satisfied in the 2-dimensional case.

L. Ahlfors (Cambridge, Mass.).

Theory of Series

Mitrinovitch, D. S. Sur les nombres de Stirling. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **1**, 49–95 (1948). (Serbian. French and Russian summaries)

Let a_r and b_r $(r=1, 2, \cdots, p)$ be arbitrary. The author finds simple expressions for

$$\sum_{k=0}^{n-1}\prod_{r=1}^{p}(a_r+(k-1)b_r).$$

He then discusses in an elementary way the difference equation $S_{n+1}^m=S_n^{m-1}-nS_n^m$ which is known from the theory of Stirling numbers.

W. Feller (Ithaca, N. Y.).

Macphail, M. S. On Perron's extension of the Euler-Knopp summation method. Trans. Roy. Soc. Canada. Sect. III (3) **42**, 43–49 (1948).

Let $\mathcal{E}_r$ denote the series-to-series transformation

$$A_n = r\sum_{k=0}^{n}\binom{n}{k}r^k(1-r)^{n-k}a_k,$$

by which the series $\sum a_n$ is evaluable to s if $\sum A_n$ converges to s. Let E_r denote the sequence-to-sequence transformation

$$\sigma_n = \sum_{k=0}^{n}\binom{n}{k}r^k(1-r)^{n-k}s_k,$$

by which a sequence s_n is evaluable to s if $\lim \sigma_n = s$. As was pointed out by Perron [Math. Z. **18**, 157–172 (1923)], the transformation $\mathcal{E}_r$ can be generated from the function $F(t) = rt/\{1-(1-r)t\}$ in the following way. The series $\sum a_n[F(t)]^{n+1}$, when formally arranged as a power series in t, becomes $\sum A_n t^{n+1}$ and this becomes $\sum A_n$ when $t=1$. Perron, being interested in regular methods only, considered general methods analogous to $\mathcal{E}_r$ generated by functions $f(t) = \sum c_n t^{n+1}$ having suitable properties and showed that the conditions $c_n \geqq 0$, $f(1)=1$ are sufficient for regularity. The author considers nonregular as well as regular methods, and gives conditions which imply that the method generated by $\sum c_n t^{n+1}$ is consistent with all other methods of the same type and, in particular, with convergence. Further theorems involving evaluability by the more general methods and the methods $\mathcal{E}_r$ are proved. The proofs are function-theoretic. When $|1-r|<1$, the theorems about $\mathcal{E}(r)$ yield theorems about the methods $E(r)$ because $\mathcal{E}(r)$ and $E(r)$ are then known to be equivalent.

R. P. Agnew (Ithaca, N. Y.).

Mambriani, A. Moltiplicazione integrale e moltiplicazione sommatoria. Atti Sem. Mat. Fis. Univ. Modena **1**, 135–142 (1947).

The author considers an algebra in which the "product" of two functions $f(x, y)$ and $g(x, y)$ is defined to be the function $\int_\alpha^\beta f(x, t)g(t, y)dt$, α and β being given fixed functions of x and y. This multiplication is distributive with respect to ordinary addition, but is in general neither commutative nor associative. Familiar special cases of this product operation are mentioned, among them cases in which the multiplication becomes associative, or associative and commutative. Analogous questions are considered for the "product" of two sequences $\{a_{mn}\}$, $\{b_{mn}\}$.

W. Strodt (New York, N. Y.).

Wall, H. S. Convergence of continued fractions in parabolic domains. Bull. Amer. Math. Soc. **55**, 391–394 (1949).

The continued fraction $1/1+c_1/1+c_2/1+\cdots$ converges if and only if some c_p vanishes, or $c_p \neq 0$ and the series $\sum |d_p|$ diverges, where $d_1 = 1$, $d_{p+1} = 1/c_p d_p$, $p = 1, 2, \cdots$. The c_p are complex numbers such that

$$|c_p| - \Re(c_p e^{i(\phi_p+\phi_{p+1})}) \leqq 2r\cos\phi_p\cos\phi_{p+1}(1-g_{p-1})g_p,$$

where r, ϕ_p, g_p are real numbers, $0<r<1$, $-\frac{1}{2}\pi+c \leqq \phi_p \leqq \frac{1}{2}\pi-c$ $(0<c<\frac{1}{2}\pi)$, $0 \leqq g_{p-1} \leqq 1$. [Cf. theorems stated by the author, Analytic Theory of Continued Fractions, Van Nostrand, New York, 1948, chapter VII; these Rev. **10**, 32].

E. Frank (Chicago, Ill.).

Wall, H. S. Note on a periodic continued fraction. Amer Math. Monthly **56**, 96–97 (1949).

Let $f_1, f_2, f_3, \cdots$ be the sequence of approximants of the continued fraction

$$1+\frac{a}{1}+\frac{a}{1}+\cdots,$$

where a is complex or else real and not less than $-\frac{1}{4}$. Let further $U_1(x), U_2(x), \cdots$ be the sequence of approximants of the continued fraction

$$x-\frac{x^2-x-a}{2x-1}-\frac{x^2-x-a}{2x-1}-\cdots.$$

Then $U_n(f_n) = f_{nn}$.

E. Bodewig (The Hague).

Fourier Series and Generalizations, Integral Transforms

Pagni, Mauro. Sui coefficienti di Fourier di una funzione di funzione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 363–368 (1948).

The author shows that the Fourier coefficients of the function of a function $f(x(t))$ can, under various simple restrictions upon $x(t)$ and $f(x)$, be expressed in terms of the Fourier coefficients of $f(x)$ and the moments, $\int_0^{2\pi} x(t)^n dt$. As an application of the method, he derives the Fourier coefficients of the inverse function $f^{-1}(x)$ in terms of those of $f(x)$.

R. Bellman (Stanford University, Calif.).

Hardy, G. H., and Rogosinski, W. W. Notes on Fourier series. V. Summability (R_1). Proc. Cambridge Philos. Soc. **45**, 173–185 (1949).

[For note IV cf. the same Proc. **43**, 10–25 (1947); these Rev. **8**, 376.] A numerical sequence $s_0, s_1, \cdots$ is said to be summable R_1 to limit s if the series (*) $(2/\pi)\sum n^{-1}s_n \sin nh$ converges for small $|h|$ and if its value tends to s as $h \to +0$. This method has some resemblance to the method $(R, 1)$ (though they are not "comparable") defining the sum of a series $\sum u_n$ as $\lim_{h\to 0}\sum u_n(nh)^{-1}\sin nh$. The main result of the paper is that, if $s_n = s_n(\theta)$ is the Fourier series of an $f \in L^r$, $r>1$, then $(R_1)s_n(\theta) \to f(\theta)$ almost everywhere. This holds even if only $|f|\log^+|f|$ is integrable or, still more generally, if the series conjugate to the Fourier series of f is itself a Fourier series. On the other hand, there is an $f \in L$ such that $s_n(\theta)$ is nowhere summable R_1, since for such an f no matter what is θ the series (*) diverges for arbitrarily small h. More precise results are also proved giving necessary and sufficient conditions that the Fourier series be summable R_1 at an individual point.

A. Zygmund.

Sinvhal, S. D. On Riesz summability of Fourier series. Proc. Benares Math. Soc. (N.S.) **8**, no. 1, 11–24 (1946).

The author gives necessary and sufficient conditions that the Fourier series of an $f(x)$ be summable (R, λ, k) at a Lebesgue point x_0 of f, provided $\lambda(w)$ satisfies certain regularity conditions. Corresponding results are obtained for the conjugate series, and in the case when

$$\int_0^t |f(x_0+t)+f(x_0-t)-2f(x_0)|dt = o\{t/|\log t|\}$$

as $t \to +0$.

A. Zygmund (Chicago, Ill.).

Davydov, N. A. The convergence of lacunary trigonometric series. Doklady Akad. Nauk SSSR (N.S.) **65**, 9–12 (1949). (Russian)

Let $\sum_1^\infty a_k\zeta^{\lambda_k}=f(\zeta)$ be a power series convergent for $|\zeta|<1$ and satisfying the conditions $a_k\to 0$, $\lambda_{k+1}/\lambda_k>q>1$. A necessary and sufficient condition that the series converge at $\zeta=1$ to sum s is that $\lim f(\zeta_n)=s$, where ζ_n is any sequence of points in $|\zeta|<1$ converging to $\zeta=1$ in the angle between two chords through that point and such that $1-\lambda_n^{-1}\leqq|\zeta_n|\leqq 1-\lambda_{n+1}^{-1}$. [Even in a more general case the result was proved by Landau, Monatsh. Math. Phys. **18**, 8–28 (1907), in particular, pp. 18–19.] *A. Zygmund.*

Bochner, S. On spherical partial sums of multiple Fourier series. Revista Ci., Lima **50**, 85–104 (1948).

Let

$$S_R^\delta(x)=\sum_{\nu\leqq R}(1-\nu^2/R^2)^\delta A_{\nu_1\dots\nu_k}(x),$$

with $\nu=(\nu_1^2+\cdots+\nu_k^2)^{\frac{1}{2}}$, be the spherical Riesz sums of the Fourier series $\sum A_{\nu_1\dots\nu_k}(x)$ of a function $f(x)=f(x_1,\cdots,x_k)$ of period 2π in each variable. For $\delta>\frac{1}{2}(k-1)$, Bochner [Trans. Amer. Math. Soc. **40**, 175–207 (1936)] gave a formula expressing $S_R^\delta(x)$ in terms of the "spherical averages" of f. The integral in this formula diverges for $0\leqq\delta\leqq\frac{1}{2}(k-1)$, and it is now shown that the formula preserves its validity even in that case, if we apply to the integral the Abel-Gauss method of summation. *A. Zygmund* (Chicago, Ill.).

Cheng, Min-Teh. Riesz summation of multiple Fourier series by spherical means. Ann. of Math. (2) **50**, 356–384 (1949).

The author studies spherical summation of multiple Fourier series first introduced by Bochner [Trans. Amer. Math. Soc. **40**, 175–207 (1936)]. Some theorems of Bochner are refined by considering a logarithmic scale of summability. Results are also obtained for convergence factors in spherical summation. *A. Zygmund* (Chicago, Ill.).

Nikol'skiĭ, S. M. The Fourier series of functions with derivatives of bounded variation. Doklady Akad. Nauk SSSR (N.S.) **65**, 13–15 (1949). (Russian)

If $f(x)$ is of period 2π and has an rth derivative $f^{(r)}(x)$ of bounded variation with discontinuities at $x_1, x_2, \cdots$, then the nth partial sum $s_n(x, f)$ of the Fourier series of f satisfies the relation

$$\|f-s_n\|_{L_p}\sim\gamma_r^{(p)}n^{-(r+1/p)}(\sum|\sigma_k|^p)^{1/p},\qquad 1<p\leqq+\infty,$$

where σ_k is the jump of $f^{(r)}(x)$ at x_k, and $\gamma_r^{(p)}$ is a constant given by a rather complicated double integral. An inequality is given for $\|f-s_n\|_{L_p}$ if $f^{(r)}$ is continuous and has a modulus of continuity $\omega(\delta)$. Finally, corresponding results are obtained for $p=1$, when the right-hand sides of the relations acquire the factor $\log n$. *A. Zygmund.*

Stečkin, S. B. On the degree of best approximation to continuous functions. Doklady Akad. Nauk SSSR (N.S.) **65**, 135–137 (1949). (Russian)

It is known that for the problems of best approximation of continuous functions $f(x)$ by polynomials the notion of the modulus of continuity $\omega(\delta, f)=\max_{x,\,|h|\leqq\delta}|f(x+h)-f(x)|$ is insufficient, and that we have to consider the higher moduli

$$\omega_k(\delta, f)=\max_{x,\,|h|\leqq\delta}\Big|\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(x+ih)\Big|,$$

especially the case $k=2$ [see Zygmund, Duke Math. J. **12**, 47–76 (1945); these Rev. **7**, 60]. Let $\varphi(\delta)$ be a positive function defined for $0<\delta\leqq\pi$, tending to 0 with δ, non-decreasing but such that $\varphi(\delta)\delta^{-\alpha}$ is decreasing for some $\alpha>0$ (the latter condition may be replaced by a weaker one). Let $\varphi(u)\sim\psi(u)$ mean that the ratio $\varphi(u)/\psi(u)$ is contained between two positive constants. The author states that if $f(x)$ is continuous and of period 2π, and if $E_n[f]$ is the best approximation of f by trigonometric polynomials of order n, then (*) $E_n[f]\sim\varphi(n^{-1})$ implies (**) $\omega_k(\delta, f)\sim\varphi(\delta)$ for every $k>\alpha$; conversely, if (**) is satisfied for some $k>\alpha$, then (*) holds. *A. Zygmund* (Chicago, Ill.).

Lozinskiĭ, S. M. Theorems on resonance and linear trigonometric polynomial operations in certain functional spaces. Doklady Akad. Nauk SSSR (N.S.) **64**, 453–456 (1949). (Russian)

[The topic of the present paper is close to that of the author's papers, same Doklady (N.S.) **59**, 1389–1392 (1948); **60**, 961–964 (1948); **61**, 193–196 (1948); these Rev. **10**, 188; and references given there.] No proofs are given. The following is one of the results stated. Let $\omega(u)$ be a continuous increasing function of $u\geqq 0$, with $\omega(0)=0$, $\lim_{u\to 0}u/\omega(u)=0$. Let E be a functional space whose elements are real-valued functions $f(t)$, $-\infty<t<\infty$, with the usual definition of addition and multiplication by a constant. Let $f^\alpha(t)$ be defined as $f(t+\alpha)$ and suppose that $\|f^\alpha\|\leqq K\|f\|$, where the constant K may depend on E but not on f or α. Let E_ω be the set of the $f\in E$ such that $\sup_{0<h<\infty}\|f^h-f\|_E/\omega(h)<+\infty$. Suppose we have a sequence $f_n\in E$ and a sequence of positive numbers $\epsilon_n\to 0$ such that $\|f_n\|\leqq\omega(\epsilon_n)$, $\|f_n^h-f_n\|\leqq\epsilon_n^{-1}\omega(\epsilon_n)h$ $(0<h<\infty)$. Let $\mathfrak{E}$ be a space of type (B), $\{U_n\}_{n=1}^\infty$ a sequence of linear operators from E to $\mathfrak{E}$ and U a linear operation from E to $\mathfrak{E}$. Then if $\limsup_n\|U_n(f_n)\|_{\mathfrak{E}}>0$ there is an $f\in E_\omega$ such that $\limsup_n\|U(f)-U_n(f)\|_{\mathfrak{E}}>0$.

A. Zygmund (Chicago, Ill.).

Remez, E. Ya. On mean, uniform (Chebyshevian) and quasiuniform approximations. Doklady Akad. Nauk SSSR (N.S.) **60**, 199–202 (1948). (Russian)

The author calls the numerical functions $v_0, v_1, \cdots$ in the space E metrically independent if, for any c_i $(\sum|c_i|>0)$, $E_1=E_x[\sum c_iv_i(x)=0]$ implies $\mu(E_1)<\mu(E)$. Lemma 1. In the above case it is possible to find g, independent of the choice of the c_i's, such that $\mu(E_1)\leqq g<\mu(E)$. Lemma 2. If in the above case $\max_i|c_i|=1$ then to every ϵ, $0<\epsilon<\mu(E)-g$, there exists an $\eta>0$ such that $\mu(E_x[|\sum c_iv_i(x)|<\eta])\leqq g+\epsilon$. In the following the author fixes $c_0=1$ and defines

$$\delta_m(\phi)=[(\mu E)^{-1}\textstyle\int_E|\phi(x)|^m dx]^{1/m},\quad \delta_0^*(\phi)=\operatorname{ess\,max}|\phi(x)|$$

for $x\in E$, $\delta_0(\phi)=\max|\phi(x)|$ for $x\in E$, where $\phi=v_0+\sum c_iv_i$. Suppose ϕ_m is the minimizing function for δ_m, ϕ_0^* for δ_0^* and ϕ_0 for δ_0; the author proves the following extension of a theorem proved by Pólya [C. R. Acad. Sci. Paris **157**, 840–843 (1913)] and Jackson [Trans. Amer. Math. Soc. **22**, 117–128 (1921)]: $\lim_{m\to\infty}\delta_0^*(\phi_m)=\delta_0^*(\phi_0^*)$. If all v_i's are almost everywhere bounded, $\lim_{m\to\infty}\delta_0(\phi_m)=\delta_0(\phi_0)$.

František Wolf (Berkeley, Calif.).

Bezlyudnyĭ, A. S. The approximation of periodic functions of two variables by interpolatory trigonometric polynomials. Doklady Akad. Nauk SSSR (N.S.) **65**, 257–260 (1949). (Russian)

Let $H^{\alpha\beta}$ be the class of functions $f(x, y)$ of period 2π in each variable and satisfying

$$|f(x_2, y_2)-f(x_1, y_1)|\leqq M|x_2-x_1|^\alpha+N|y_2-y_1|^\beta,$$
$$|\Delta(f, x_1, x_2, y_1, y_2)|\leqq c|x_2-x_1|^\alpha|y_2-y_1|^\beta,$$

where

$$\Delta(f, x_1, x_2, y_1, y_2) = f(x_2, y_2) + f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1).$$

Let $S_{mn}(x, y)$ denote the interpolating polynomial of orders m, n in x and y, respectively, coinciding with $f(x, y)$ at the points $(2k\pi/(2m+1),\ 2l\pi/(2n+1))$, $k=0, \pm1, \cdots, \pm m$; $l=0, \pm1, \cdots, \pm n$. It is shown that

$$\sup_{f\in H^{\alpha,\beta}} |f(x,y) - S_{mn}(x,y)| = \frac{M \log m}{\pi^{1-\alpha} m^{\alpha}} |\sin (m+\tfrac{1}{2})x| + \frac{N \log n}{\pi^{1-\beta} n^{\beta}} |\sin (n+\tfrac{1}{2})y| + O(m^{-\alpha}) + O(n^{-\beta}) + O\left(\frac{\log m \cdot \log n}{m^{\alpha} n^{\beta}}\right).$$

A. Zygmund (Chicago, Ill.).

Levin, B. A new construction of Levitan's theory of almost periodic functions. Doklady Akad. Nauk SSSR (N.S.) **62**, 585–588 (1948). (Russian)

B. Levitan generalized the concept of an almost periodic function in the following way [Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **15**, no. 2, 3–34 (1938)]. An L.a.p. function will be such that (i) it is continuous, (ii) for every ϵ, $N>0$, there exists a relatively dense set of intervals $E_{\epsilon N}$, such that $\tau \in E_{\epsilon N}$ and $|x| \leqq N$ implies $|f(x+\tau)-f(x)|<\epsilon$, (iii) to an arbitrary $E_{\epsilon N}$ there exist $\eta, \rho>0$ such that $\tau' \in E_{\epsilon N}$ and $\tau'' \in E_{\epsilon N}$ imply $\tau' \pm \tau'' \in E_{\epsilon N}$. The present author gives an equivalent definition of an L.a.p. class which can be easily extended from the real axis to an arbitrary group. Define $\{\tau_n\} \xrightarrow{f} \tau_0$ to be equivalent to $\lim_n f(x+\tau_n-\tau_0)=f(x)$ for all x, and $\{\tau_n\} \xrightarrow{f} 0$ to be equivalent to $\lim_{m,n} f(x+\tau_m-\tau_n)=f(x)$ for all x. This introduces a new topology on the real axis. The author announces the following theorems. (1) A necessary and sufficient condition for f to be L.a.p. is that (i) f is continuous, (ii) the above defined topology makes the real axis conditionally compact, (iii) $\tau_n \xrightarrow{f} 0$, $\tau_n' \xrightarrow{f} 0$ imply $\tau_n + \tau_n' \xrightarrow{f} 0$. The last condition implies the f-continuity of addition. Hence by the new topology the real axis is turned into a conditionally compact topological group Ω_f. Call its completion T_f. Then "$\tau_n \xrightarrow{f} \tau_0$ implies $f(\tau_n) \to f(\tau_0)$" shows that f as a function on T_f is continuous. It is easy to deduce that any uniformly continuous function on Ω_f corresponds to a Bohr a.p. function on the real axis. Every continuous character of T_f generates a character $e^{i\lambda x}$; the numbers λ thus obtained form a modul which the author calls the modul of the L.a.p. function f. (2) The statement "φ is an L.a.p. function with a modul contained in that of f" is equivalent to "φ is continuous on Ω_f and $f(x+\tau_n) \to f(x)$ implies $\varphi(x+\tau_n) \to \varphi(x)$ for all x." The author defines integration over T_f by $\int_{T_f} \varphi(p)dp = \lim_{T\to\infty} \int_{-T}^{T} \varphi(x)dx$, and defines Fourier series and approximation in the natural way.

František Wolf (Berkeley, Calif.).

Purificato, Angela. Su un particolare sistema di funzioni ortogonali e su un procedimento di sommazione analogo a quello di Poisson. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 348–355 (1948).

For fixed c the sequence $[f_n^{(c)}(x)]$ of functions

$$f_0(x) = e^{cx}[2c/(e^{2c\pi}-1)]^{\frac{1}{2}},$$
$$f_n(x) = (n^2+c^2)^{-\frac{1}{2}} \cdot (c \sin nx + n \cos nx)(2/\pi)^{\frac{1}{2}}$$

is orthonormal in the interval $(0, \pi)$. The Fourier expansion of a function $F(x)$ in this system ($c \neq 0$) generalizes the Fourier cosine series, to which it reduces for $c=0$. The author proves that the classical theorems of Fejér and Lebesgue on the $(C, 1)$ summability of Fourier trigonometric series by arithmetic means of first order hold also for the system $[f_n(x)]$. With the aid of Bromwich's theorem it is proved further that, for a series $\sum_0^\infty u_n$ summable with the sum s by the arithmetic means of first order,

$$\lim_{t\to 0} \sum_{n_0+1}^{\infty} u_n \cdot e^{-t(n^2-k^2)^{\frac{1}{2}}} = s - \sum_0^{n_0} u_m,$$

where $n_0 \leqq k < n_0+1$, if the fixed parameter k is positive, and $n_0=0$ otherwise.

E. G. Kogbetliantz (New York, N. Y.).

Ilieff, Lubomir. Sur une classe de fonctions à zéros réels. C. R. Acad. Bulgare Sci. Math. Nat. **1**, no. 2–3, 15–18 (1948).

Let $f_0(t)$ be even, nonnegative and integrable in $(-1, 1)$, and such that the polynomials $P_n(f_0, z) = \int_{-1}^{1} f_0(t)(1+itz)^n dt$, $n=2, 3, \cdots$, have only real zeros. Let q be a positive integer. Put $x=1-t^{2q}$, $f_0(t)=\varphi_0(x)$. Then $f_1(t)=\varphi_1(x)=\int_0^x \varphi_0(t)dt$ has the property that the polynomials $P_n(f_1, z)$ have only real zeros; if the zeros of $P_n(f_0, z)$ are, in addition, simple, so are those of $P_n(f_1, z)$. Since $P_n(f, z/n)$ converges to $\int_{-1}^{1} f(t) \cos tz\, dt$, the latter integral has the same properties for its zeros. In particular, starting from a decreasing function $\varphi_0(x)$ in $(0, 1)$, form the function $f_0(t)=\varphi_0(x)$, $x=1-t^{2q}$; then $f_0(t)$ has the property in question. This leads to a class of symmetrically decreasing functions $f(t)$ such that $\int_{-1}^{1} f(t) \cos tz\, dt$ has only real zeros.

R. P. Boas, Jr. (Providence, R. I.).

Krishnan, K. S. On the equivalence of certain infinite series and the corresponding integrals. J. Indian Math. Soc. (N.S.) **12**, 79–88 (1948).

This paper contains more details, with several specific examples, of work previously reported [Bhatia and Krishnan, Proc. Roy. Soc. London. Ser. A. **192**, 181–194 (1947); Krishnan, Nature **162**, 215 (1948); these Rev. **10**, 35].

R. P. Boas, Jr. (Providence, R. I.).

Geronimus, Ya. L. On the closure of certain systems of functions in the space L_w^p. Doklady Akad. Nauk SSSR (N.S.) **64**, 611–614 (1949). (Russian)

Let C be a simple closed rectifiable curve in the w-plane bounding a domain D, with $0 \in D$. The parameter representation of C is $w=\xi(s)$ ($0 \leqq s < l$; s=arc length). Let L_w^p be the space of complex-valued functions $f(\xi)=f(\xi(s))$ defined on C and with norm $\|f\| = (\int_0^l |f(\xi)|^p d\sigma(s))^{1/p} < \infty$ ($p \geqq 1$, $\sigma(s)$ bounded, nondecreasing). Let $\{\varphi_n^{\alpha}(w)\}$ (α fixed, $n=0, 1, 2, \cdots$) be functions regular in a domain $D' \supset D$ and such that every function regular in $D+C$ can be expanded in a uniformly convergent series $\sum a_n \varphi_n^{\alpha}$. The closure in L_w^p of $M=\{f_\alpha(\xi)\varphi_n^{\alpha}(\xi)\} + \{\psi_\beta(\xi)\} \subset L_w^p$ is considered (α, β run through finite or infinite sets of indices). Let $w=\psi(z)$ map $|z|<1$ on D, and let $z=\gamma(w)$ be the inverse function of ψ. Say that $f(w) \in E$, if f is regular in D and $\int_{|z|=r} |f(\psi(z))\psi'(z)||dz| = O(1)$ ($r<1$). Let f_0 be any one of the functions f_n. The following theorems are given.

(I) The set M is not closed if and only if there is a function $\lambda_0(w)\varepsilon E$ whose boundary values $\lambda_0(\xi)$ satisfy

$$(1)\qquad \int_C |\lambda_0(\xi)/f_0(\xi)(\sigma(s))^{1/p}|^q|d\xi| < \infty, \quad q=p/(p-1),$$

$$(2)\qquad \int_C \lambda_0(\xi)\psi_\beta(\xi)/f_0(\xi)d\xi = 0, \qquad \psi_\beta \varepsilon M,$$

(3) $\lambda_0(w)f_\alpha(w)/f_0(w)\varepsilon E$ for all α. [The interpretation of (1) for the case that $\sigma'(s)=0$ in a set of positive measure is not indicated.] (II) If

$$\int_C \log(\sigma'(s)|f_0(\xi)|^p)|\gamma'(\xi)||d\xi| = -\infty,$$

the set M is closed. (III) If and only if

$$\int_C \log \sigma'(s)|\gamma'(\xi)||d\xi| > -\infty,$$

the set $\{\xi^n\}$ $(n=0, 1, 2, \cdots)$ is not closed. (IV) Let N be the set of all integers with the exception of $r_1 \geqq r_2 \geqq \cdots \geqq r_m$. The set $\{\xi^n\}$ $(n\varepsilon N)$ is not closed if there is a $\lambda_0(w)=\sum_{k=1}^m \mu_k w^{r_k-r_k}$ satisfying condition (1) of theorem I with $f_0=1$. (V) If $k>0$, $p\geqq 1+k^{-1}$, and $\int_C(\sigma'(s))^{-k}|d\xi|<\infty$, then $\{\xi^n\}$ $(n=0, \pm 1, \pm 2, \cdots)$ is closed, but ceases to be closed on removal of any one ξ^n. *W. H. J. Fuchs* (Liverpool).

Boas, R. P. The completeness of some sets of analytic functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 55–60 (1949). (Russian)

The set $\{f_n(z)\}$ of functions regular in $|z|<R$ is called complete if, for every $r<R$,

$$\int_0^{2\pi} f_n(re^{i\theta})g(\theta)d\theta = 0, \qquad n=1, 2, \cdots; g\varepsilon L^2,$$

implies $\int_0^{2\pi} F(re^{i\theta})g(\theta)d\theta=0$ for every $F(z)$ regular in $|z|\leqq r$.

Theorem A. Let $f(z)$ be regular in $|y|<\pi$, of period 2π; let $2\pi c_n=\int_0^{2\pi} f(t)e^{-int}dt$; and let $\alpha_n=\beta_n+i\gamma_n$, where $\beta_n\geqq 0$, and $\{\gamma_n\}$ has a limit point in $|y|<\eta$ $(0<\eta<\pi)$ or 0 is a limit point of $\{\gamma_n\}$ $(\eta=0)$. Let (A) $c_{n_p}\neq 0$ for a sequence $\{n_p\}$ of lower density greater than ζ/π, $\zeta<\pi-\eta$. Then $\{f(z+\alpha_n)\}$ is complete in $|z|<\zeta$. Sketch of proof. There is no loss of generality in supposing $|\gamma_n|<\pi-\zeta$. The function

$$(*)\qquad h(z)=\int_0^{2\pi} f(re^{i\theta}+z)g(\theta)d\theta=\sum_{-\infty}^{\infty} c_n e^{inz}\int_0^{2\pi} e^{inw}g(\theta)d\theta$$

$(w=re^{i\theta},\ r<\zeta)$ vanishes identically, since $h(\alpha_n)=0$. Hence $\int_0^{2\pi} e^{in_p w}g(\theta)d\theta=0$. But $\{e^{in_p w}\}$ is complete in $|w|<s<\zeta$ [Marcouchewitch, Rec. Mat. [Mat. Sbornik] N.S. **17**(**59**), 211–252 (1945); these Rev. **7**, 425] and the completeness of $\{f(z+\alpha_n)\}$ follows.

Theorem B. Let $f(z)$ be an integral function of order $\rho>1$, type σ, of period 2π; let $\delta_n=\beta_n+i\gamma_n$, where $0\leqq\beta_n<2\pi$, $|\gamma_{n+1}|\geqq|\gamma_n|$. If the condition (A) of theorem A is satisfied, then $\{f(z+\delta_n)\}$ is complete in $|z|<\zeta<\pi$, provided that

$$\limsup_{n\to\infty} |\gamma_n|^{\rho-1}/n < 1/2\sigma\rho.$$

The proof consists in showing by means of Jensen's formula that (*) vanishes identically and then proceeding as in theorem A.

Theorem C. Let $f(z)=\sum c_k z^k$ be regular and bounded in $|z|<1$ and at $z=1$. Let $\alpha_n=\beta_n+i\gamma_n$, $\beta_n\to\infty$. Then $\{f(x^{\alpha_n})\}$ is closed in $L^2(0, 1)$ if $\sum\beta_k/|\alpha_n|^2=\infty$. The proof follows the method of a proof of Korevaar [Duke Math. J. **14**, 31–50 (1947); these Rev. **9**, 29]. *W. H. J. Fuchs.*

Kaushik, S. P. A theorem on the generalised Laplace's transform. Proc. Benares Math. Soc. (N.S.) **8**, no. 1, 7–10 (1946).

If

$$\phi_{k,m}(p)=p\int_0^\infty (2xp)^{-\frac{1}{2}}W_{k,m}(2xp)f(x)dx$$

and

$$\psi(p)=p\int_0^\infty e^{-xp}\phi_{k,m}(x)dx,$$

the author evaluates ψ in terms of f by changing the order of integration. The example $f(x)=x^{n-\frac{1}{2}}J_{n-\frac{1}{2}}(ax)$ is discussed.

A. Erdélyi (Pasadena, Calif.).

Polynomials, Polynomial Approximations

***Erim, Kerim. Ein algebraisches Theorem.** Université d'Istanbul. Faculté des Sciences. Recueil de mémoires commémorant la pose de la première pierre des Nouveaux Instituts de la Faculté des Sciences, pp. 33–38, Istanbul, 1948.

The author proves the theorem that the equation

$$a_0+a_1z+\cdots+a_nz^n=0$$

has k zeros $(k\leqq n)$ in the unit circle if $|a_k|$ exceeds the sum of the moduli of the other coefficients. This is proved by use of Rouché's theorem. By a linear transformation, a corresponding theorem for a half-plane is derived. If $k=n=3$, the hypothesis on the a_k implies that the point $(a_0/a_3, a_1/a_3, a_2/a_3)$ lies in the octahedron with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. [Reviewer's note. The above theorem is a special case of Pellet's theorem.]

M. Marden (Milwaukee, Wis.).

Herrmann, Aloys, et Souriau, Jean-Marie. Un critère de stabilité déduit du théorème de Sturm. C. R. Acad. Sci. Paris **228**, 1183–1184 (1949).

Let p, q and r denote respectively the number of zeros of a polynomial $P(x)$ in the upper half plane, on the real axis and in the lower half plane. For x real, let $P(x)=P_1(x)+iP_2(x)$ and let $V(x)$ denote the number of variations of sign in the corresponding Sturm sequence $P_1(x), P_2(x), P_3(x), \cdots, P_k(x)$. The authors point out that the difference $p-r$ may be found from the increment in $\arg P(x)$ along the real axis and hence $p-r=V(+\infty)-V(-\infty)$. They infer as a necessary and sufficient condition for $r=0$, and thus for stability, that the leading coefficients of the $P_j(x)$, $j=1, 2, \cdots, k$, have alternating signs. [Reviewer's note. This is a well-known procedure.] *M. Marden* (Milwaukee, Wis.).

Egorova, I. A. On the principle of localization in the theory of interpolation. Doklady Akad. Nauk SSSR (N.S.) **64**, 445–447 (1949). (Russian)

Let $L_n(f, x)=\sum_{k=1}^n l_k^{(n)}(x)f(x_k^{(n)})$ be the Lagrange interpolating polynomial of a function $f(x)$. Here $x_k^{(n)}$ and $l_k^{(n)}(x)$ are the fundamental points and the fundamental polynomials of the interpolation $(k=1, 2, \cdots, n; n=1, 2, \cdots)$. We say that the system $\{x_k^{(n)}\}$ satisfies the principle of localization at a point x_0, if $L_n(f, x_0)\to 0$ for any function f integrable R and vanishing in a neighborhood of x_0. Let $[A, B]$ be an interval containing all the $x_k^{(n)}$ and $[a, b]$ an arbitrary subinterval of $[A, B]$ not containing x_0. The following result is stated without proof. A necessary and

sufficient condition that the system $\{x_k^{(n)}\}$ of fundamental points satisfy the principle of localization at x_0 is that (1) $\sum l_k^{(n)}(x_0)\to 0$ as $n\to\infty$ (all summations over $x_k^{(n)}\in[a,b]$); (2) $\sum|l_k^{(n)}(x_0)|<M$ for all n; (3) if

$$E(I)=\limsup_{n\to\infty}\sum|l_k^{(n)}(x_0)|$$

for any set I consisting of a finite number of nonoverlapping intervals, then $E(I_i)\to 0$ for every sequence $I_1\supset I_2\supset\cdots$ such that $x_0\bar{\in}I_1$ and meas $I_i\to 0$. *A. Zygmund.*

Martin, Yves. Sur une classe de séries de polynomes. C. R. Acad. Sci. Paris **228**, 1195–1196 (1949).

Several theorems are stated without proof concerning the convergence of the series of interpolation $\sum_{n=1}^{\infty}a_n p_n(z)$, $p_n(z)=\prod_{i=1}^{n}(1-z/\lambda_i)$, $0<\lambda_1<\lambda_2<\cdots$, $\lambda_n\to\infty$, as it relates to the convergence of the series $\sum_{n=1}^{\infty}1/\lambda_n$.

E. N. Nilson (Hartford, Conn.).

Special Functions

***Magnus, Wilhelm, and Oberhettinger, Fritz. Formulas and Theorems for the Special Functions of Mathematical Physics. Translated by John Wermer.** Chelsea Publishing Company, New York, N. Y., 1949. viii+172 pp. $3.50.

The German original was published by Springer, Berlin, 1943; these Rev. **9**, 183. A second German edition appeared in 1948; these Rev. **10**, 38.

***Tricomi, Francesco. Elliptische Funktionen. Translated and edited by Maximilian Krafft.** Mathematik und ihre Anwendungen in Physik und Technik, Reihe A, Band 20. Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1948. ix+315 pp.

[The Italian original was published by Zanichelli, Bologna, 1937.] This is an extremely well written book on the generally useful parts of the theory of elliptic functions. In a subject which, for a century and a half, has been characterized by an overpowering mass of formal relations and identities, the author has successfully avoided succumbing to the temptation of trying to amaze the reader by surprising identities and of including formulas for their sheer beauty. Some other authors have tried to overcome the embarras de richesse by solving an "extremal problem," namely by deriving a maximum of relations in a minimum of space. The author has steered clear of this temptation. The book is well-organized and at no place is coherence sacrificed in favor of a mere display of ingenuity. Derivations which are inessential from the overall point of view of the book are strictly left alone.

Much care has been devoted to clarity of exposition. In contradistinction to a widespread custom in mathematical literature, statements like "it is easily seen" are invariably followed up by full explanations. While it is in the nature of things that the knowledge of some results in the elementary theory of functions of a complex variable has to be assumed, the amount of presupposed knowledge in that field is kept at a minimum; where feasible, such auxiliary results are proved in the text.

A particularly valuable feature of the book is a chapter on applications of elliptic functions which contains material otherwise not easily accessible. It is also worthy of note that the author takes the term "applications" seriously and does not hesitate to carry through the ensuing computations to the last numerical detail. This book can be strongly recommended to the students of both pure and applied mathematics. *Z. Nehari* (St. Louis, Mo.).

Reed, I. S. Note on a minimum value of the gamma function. Revista Ci., Lima **50**, 151–154 (1948).

An infinite series is given for the point z_1 on the real axis at which $|\Gamma(z)|$ attains its minimum value for $1\leqq\Re(z)\leqq 2$. *L. Schoenfeld* (Urbana, Ill.).

Goodwin, E. T. Recurrence relations for cross-products of Bessel functions. Quart. J. Mech. Appl. Math. **2**, 72–74 (1949).

With

$$p_n=J_n(x)\,Y_n(y)-J_n(y)\,Y_n(x),\quad q_n=\partial p_n/\partial y,$$
$$r_n=\partial p_n/\partial x,\quad s_n=\partial^2 p_n/\partial x\partial y,$$

the author finds

$$p_{n+1}-p_{n-1}=-2n(q_n/x-r_n/y),$$
$$q_{n+1}+r_n=np_n/x-(n+1)p_{n+1}/y,$$
$$r_{n+1}+q_n=np_n/y-(n+1)p_{n+1}/x,$$
$$s_n=\tfrac{1}{2}p_{n+1}+\tfrac{1}{2}p_{n-1}-n^2p_n/(xy),$$
$$p_ns_n-q_nr_n=4/(\pi^2xy).$$

The corresponding relations are obtained also for the cross products in which the index n is raised by unity instead of differentiation with respect to the variable, for cross products of modified Bessel functions, and for "spherical" Bessel functions. In the last case an alternative set of formulae is given for the cross products arising naturally in scattering problems. *A. Erdélyi* (Pasadena, Calif.).

Stankiewicz, Lidia. Sul calcolo della deformazione della piastra poggiata su suolo elastico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 339–344 (1948).

According to E. Volterra the problem mentioned in the title can be solved in terms of the function

$$F(x,y)=\int_0^\infty\int_0^\infty\frac{\sin xu\,\sin yv}{(u^2+v^2)^2+1}\,\frac{du\,dv}{uv}.$$

In this paper two expansions of $F(x,y)$ are given together with a numerical table, to three decimal places, for $x,y=0.0(.5)8(1)12,\ \infty$. *A. Erdélyi* (Pasadena, Calif.).

Erdélyi, A. Lamé-Wangerin functions. J. London Math. Soc. **23**, 64–69 (1948).

Verf. wendet sich jenen Laméschen Funktionen zu, welche Lösungen der bekannten Laméschen Differentialgleichung sind im Falle, dass dieselbe hervorgeht aus der Aufspaltung der Laplaceschen Differentialgleichung in krummlinigen Koordinaten, denen ein System von konfokalen Umdrehungszykliden zugrundeliegt. Verf. hat gefunden, dass in allen Fällen mit Ausnahme des abgeplatteten Ringes ein neuer Typ von Laméschen Funktionen eingeführt werden muss, der von ihm Lamé-Wangerinsche Funktionen genannt wird. In der vorliegenden Arbeit zeigt Verf. eines der Koordinatensysteme auf, welche auf diese Funktionen führen und leitet einige ihrer wichtigsten Eigenschaften ab. Für die Ableitung der Eigenwerte benutzt er insbesondere rasch konvergierende Kettenbrüche. *M. J. O. Strutt.*

Ascoli, Guido. Sopra un'estensione dell'equazione di Whittaker per le funzioni di Mathieu. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **10(79)**, 155–160 (1946).

Verf. geht aus von einer Whittakerschen Integralgleichung für die Lösungen der Mathieuschen Differentialgleichung im

periodischen Falle. Er zeigt, dass diese Integralgleichung zur Integration einer partiellen Differentialgleichung zweiter Ordnung benutzt werden kann. Es werden namentlich allgemeinere Kerne dieser Art betrachtet. *M. J. O. Strutt.*

Campbell, Robert. Sur une catégorie remarquable de solutions de l'équation de Mathieu associée. C. R. Acad. Sci. Paris **226**, 2114–2116 (1948).

In einer früheren Arbeit hat Verf. Lösungen der assoziierten Mathieuschen Differentialgleichung studiert, welche für bestimmte Werte eines Parameters Gegenbauersche Polynome sind. In der vorliegenden Arbeit studiert er Lösungen dieser Differentialgleichung, welche unter gewissen Bedingungen für die Parameter Gegenbauersche Funktionen sind. *M. J. O. Strutt* (Zürich).

Bouwkamp, C. J. A note on Mathieu functions. Nederl. Akad. Wetensch., Proc. **51**, 891–893 = Indagationes Math. **10**, 319–321 (1948).

Der Verf. fügt den bekannten Beweisen des Theorems über die Doppelpunkte der Grenzkurven zwischen labilen und stabilen Lösungsgebieten der Mathieuschen Differentialgleichung in der Parameterebene einen neuen Beweis hinzu. *M. J. O. Strutt* (Zürich).

Harmonic Functions, Potential Theory

Verblunsky, S. A theorem on positive harmonic functions. Proc. Cambridge Philos. Soc. **45**, 207–212 (1949).

Using the monotonicity of $r/(1+r^2-2r\cos\phi)$ as a function of r and the Poisson-Stieltjes integral for a positive harmonic function $h(z)$, the author proves the following theorems:

(i) $$0<\lim_{r\to 1} h(r)/(1-r)=\lim_{r\to 1}\inf h(z)/(1-r);$$

(ii) $$\lim_{r\to 1} h(r)(1-r)=\lim_{r\to 1}\sup h(z)(1-r);$$

(iii) if $\lim h(r)/(1-r)$ is finite, then $\lim h^*(z)$ is finite, where $h^*(z)$ is a harmonic function which is conjugate to $h(z)$, and if h^* is chosen so that $\lim h^*(r)=0$, then $\lim h^*(r)/(1-r)=0$. This theorem is applied to the proof of a theorem due to Carathéodory [Conformal Representation, Cambridge University Press, 1932, p. 51]. If $f(z)$ is regular and $|f(z)|<1$ in the unit circle, and there is a sequence z_n such that $z_n\to 1$, $f(z_n)\to 1$, $\lim (1-|f(z_n)|)/(1-|z_n|)<\infty$, then $(1-|f(r)|)/(1-r)$, $|1-f(r)|/(1-r)$, $(1-f(r))/(1-r)$ tend to the same positive limit as $r\to 1$. Also a short proof of a theorem due to Herzig [Math. Z. **46**, 129–156 (1940), p. 149; these Rev. **1**, 213] is given. *František Wolf.*

Keller, Heinrich. Sur une valeur moyenne de certaines fonctions harmoniques de trois variables. C. R. Acad. Sci. Paris **228**, 887–888 (1949).

The author considers a function u harmonic in an unbounded domain G of three-dimensional space. He subjects G and u to certain restrictions including the requirements that T_z (the region of the plane z=constant made up of interior points of G) is contained in a sufficiently large circle, and that u vanishes continuously, and has continuous first partial derivatives, at all finite boundary points of G. Let

$$m(z)=\iint_{T_z} u^2 dxdy.$$

If G is a part of the interior of a cylinder with rulings parallel to the z-axis, then on setting $k^2=\alpha^2/A$ (where α is the smallest positive root of $J_0(r)=0$ and A is the area of the normal section of the cylinder) the various possibilities concerning the behavior of $e^{\pm kz}m(z)$ as $z\to\pm\infty$ may be classified. The author also gives similar results for the case in which G is half a cone, considering the behavior as $z\to\infty$ of $z^{-p}m(z)$ and $z^{p-1}m(z)$, where p is a certain constant. He notes that these results give information concerning the growth of max $|u|$ on T_z. *F. W. Perkins.*

Kogan, S. Ya. On the solution of the Neumann problem by Schwarz's alternating method. Doklady Akad. Nauk SSSR (N.S.) **65**, 261–264 (1949). (Russian)

Let D be the union of two domains D_1 and D_2 in n-space, bounded by sufficiently smooth surfaces. The author assumes that the Neumann problem of potential theory can be solved for D_1 and D_2 and obtains a solution for D by an obvious modification of the alternating method. Since he does not prove that the whole sequence of approximating functions converges to the desired solution, he must resort to a compactness argument.

L. Bers (Princeton, N. J.).

Fichera, Gaetano. Sulla risoluzione di un particolare sistema di due equazioni vettoriali. Matematiche, Catania **3**, 10–15 = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **225** (1948).

Let D be a domain bounded by the surface F. The following vector equations are considered: (1) $\nabla^2\omega^i=\alpha^i$ in D; div $\omega^i=g$ on F. The function g initially only given on F is determined in D by the equation $\nabla^2 g=\operatorname{div}\alpha^i$. Let h be a harmonic function in the domain outside F, for which $\partial h/\partial n=(\alpha_i-\partial_i g)n^i$ and n^i is the unit normal of F. By means of these functions g and h the author is able to give a particular solution of (1). Then the general solution for the case $\alpha^i=g=0$ is derived which leads to a formula for the most general solution of (1). By this formula the problem is reduced to the determination of harmonic functions outside F. *J. Haantjes* (Leiden).

Fichera, Gaetano. Teorema d'esistenza per il problema bi-iperarmonico. Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 319–324 (1948).

L'auteur donne une méthode nouvelle simple dans l'espace à un nombre quelconque de dimensions, sous des hypothéses un peu plus générales que les usuelles. Étant donné un domaine D limité par une frontière Σ formée de quelques surfaces assez régulières, on suppose qu'il y existe une fonction H admettant des dérivées secondes à carré sommable dans D et satisfaisant sur la frontière à $H=f$, $dH/dn=g$ (f, g donnés). On cherche u biharmonique dans D avec $\Delta^2 u$ sommable et valeurs-limite f et g sur les normales pour u et du/dn. Si v_i est un système de fonctions harmoniques fondamentales déjà utilisé par l'auteur après Deny [voir Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 502–507 (1947); ces Rev. **10**, 192] et ici normalisées sur D, soit $c_i=\int_\Sigma (fdv_i/dn-gv_i)d\sigma$ et $\varphi=\sum c_i v_i$ (série convergente de somme harmonique). Alors la solution est fournie, par

exemple dans l'espace à 3-dimensions, par

$$\frac{1}{4\pi}\left\{\int_\Sigma\left[f(Q)\frac{d}{dn}\frac{1}{PQ}-g(Q)\frac{1}{PQ}\right]d\sigma-\int_D\varphi(Q)\frac{1}{PQ}dv_Q\right\}.$$

M. Brelot (Grenoble).

Fichera, Gaetano. Teoremi di completezza sulla frontiera di un dominio per taluni sistemi di funzioni. Ann. Mat. Pura Appl. (4) **27**, 1–28 = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **230** (1948).

L'auteur développe une note sans démonstrations déjà analysée [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 502–507 (1947); ces Rev. **10**, 192]. Il utilise, en les démontrant, des propriétés du potentiel de simple et double couche annoncées dans un autre mémoire déjà analysé [Giorn. Mat. Battaglini (4) **77**, 184–199 (1947); ces Rev. **10**, 298]. *M. Brelot* (Grenoble).

Haegi, H.-R. Sur le maximum du rayon intérieur. C. R. Acad. Sci. Paris **228**, 891–892 (1949).

Soit D un domaine simplement connexe dont le complémentaire contient le point à l'infini et un autre point. Faisons-en l'application conforme sur un cercle de centre 0 et rayon r_s, faisant correspondre le point $M(z)$ au point 0 avec rapport de similitude égal à 1 en 0; r_s est dit rayon intérieur de D par rapport à z [Pólya-Szegö]. L'auteur en énonce diverses propriétés: analyticité, sousharmonicité de $\log(1/r_s)$, cas d'unicité du point où r_s atteint son minimum, etc. Mais je dois souligner une propriété non mentionnée qui se trouve déjà à peu près dans A. F. Monna [Nederl. Akad. Wetensch. Proc. **45**, 43–50 (1942); ces Rev. **5**, 260]: c'est que $\log(1/r_s)$ est la valeur en M de la solution du problème de Dirichlet pour D et la donnée-frontière $\log(1/MP)$; car il semble que le vrai problème soit l'étude de cette valeur pour des domaines généraux du plan ou de l'espace ($1/MP$ remplaçant le log). *M. Brelot.*

Polya, Georges, et Meyer, Burnett. Sur les fonctions sphériques de Laplace de symétrie cristallographique donnée. C. R. Acad. Sci. Paris **228**, 1083–1084 (1949).

In an earlier note [same C. R. **228**, 28–30 (1949); these Rev. **10**, 281] the authors computed the number of linearly independent harmonic functions of order m, invariant under a finite group of orthogonal transformations. Now they express the functions themselves in the form $r^{2m+1}P(\partial/\partial x, \partial/\partial y, \partial/\partial z)r^{-1}$, where $P(x, y, z)$ is a certain combination of the polynomials xyz, $x^4+y^4+z^4$, $(y^2-z^2)(z^2-x^2)(x^2-y^2)$ and the real and imaginary parts of $(x+yi)^n$. For simplicity they have omitted the icosahedral groups. *H. S. M. Coxeter* (Toronto, Ont.).

Poritsky, H. Homogeneous harmonic functions. Quart. Appl. Math. **6**, 379–388 (1949).

Verf. betrachtet Lösungen der dreidimensionalen Laplaceschen Differentialgleichung, welche homogen vom Grade n in den Koordinaten sind. Besonderes Interesse wendet er dem Falle $n=0$ zu. Nach Transformation der Laplaceschen Differentialgleichung auf Kugelkoordinaten benutzt Verf. die Homogenitätsbeziehung um den Kugelradius zu eliminieren und erhält somit eine Differentialgleichung auf der Einheitskugel. Dann benutzt er die Homogenitätsbeziehung um aus der Differentialgleichung z zu eliminieren und erhält eine Differentialgleichung in der Ebene $z=1$. Für $n=0$ wird gezeigt, dass die Laplacesche Differentialgleichung auf der Einheitskugel durch Einführung von isothermen Koordinaten in eine zweidimensionale Laplacesche Gleichung transformiert werden kann. In diesem Falle können die Lösungen durch analytische Funktionen einer einzigen komplexen Variablen ausgedrückt werden. Auch im Falle positiver und negativer Werte von n können die Lösungen durch analytische Funktionen einer komplexen Variablen ausgedrückt werden.

M. J. O. Strutt (Zürich).

Birkhoff, Garrett, and Burton, Lindley. Note on Newtonian force-fields. Canadian J. Math. **1**, 199–208 (1949).

Les auteurs reprennent les formules classiques de Gauss et Poisson en théorie du potentiel pour les étendre au cas d'une distribution de masses pourvue d'une densité en volume satisfaisant à des conditions plus larges que celles anciennes (comme la continuité par morceaux qui se trouve remplacée par la sommabilité, par exemple), ce qui améliore des vieux résultats de Petrini. *M. Brelot* (Grenoble).

Janković, Zlatko. Deux cas simples de la théorie du potentiel. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **3**, 152–171 (1948). (Croatian. French summary)

Using the method of Fourier series, the author solves certain boundary value problems for symmetrical potential fields outside a sphere and an infinite circular cylinder.

W. Feller (Ithaca, N. Y).

Aymerich, Giuseppe. Trasformazione conforme delle funzioni biarmoniche ed applicazione all'elasticità piana. Rend. Sem. Fac. Sci. Univ. Cagliari **17** (1947), 1–12 (1948).

The author applies the theory of analytic functions of order 2 [Burgatti, Boll. Un. Mat. Ital. **1**, 8–12 (1922)] to the problem of finding a function Φ biharmonic in a domain Σ when a function ϕ biharmonic in a conformal image σ is known. He gives a brief solution of this problem. Let $Z=X+iY$ be the function of $z=x+iy$ which maps σ conformally upon Σ. Let $f(z)$ be an analytic function of order 2; then if $f=\phi(x, y)+i\psi(x, y)$, ϕ and ψ are conjugate biharmonic functions, and f may be put into the form $f=\bar{z}g(z)/2+h(z)$, where g and h are analytic. Let the analytic functions G and H be given by $G(Z)\equiv g(z)$, $H(Z)\equiv h(z)$. Then $F\equiv\bar{Z}G(Z)/2+H(Z)$ defines a one-to-one correspondence between functions f which are analytic of order 2 in σ and functions F which are analytic of order 2 in Σ. The author gives real expressions for the corresponding conjugate biharmonic functions Φ and Ψ in terms of x, y, X, Y, ϕ, ψ. He shows that Levi-Civita's inversion formulae are an immediate special case of his result. He shows that if ϕ is the Airy stress function for a plane elastic problem in σ, then Φ is the Airy stress function for a corresponding problem in Σ. He applies this result to derive from the known general solution for a complete annular region a general solution for an annular sector of arbitrary angle.

C. Truesdell (Washington, D. C.).

Duffin, R. J. On a question of Hadamard concerning super-biharmonic functions. J. Math. Physics **27**, 253–258 (1949).

On sait qu'une plaque élastique encastrée soumise à une pression normale $\rho(x, y)$ subit un déplacement w, nul au bord ainsi que les dérivées premières, et satisfaisant $a^2\Delta^2w=\rho$. Hadamard s'est demandé si $\rho\geqq 0$ entraîne $w\geqq 0$. L'auteur

étudie la question pour des rectangles. Dans le cas d'une bande infinie avec pression $p(x)$ (constante transversalement), l'usage de l'intégrale de Fourier permet une solution explicite d'où résulte que si $p(x)>0$, nul hors d'un intervalle moindre que la largeur de la bande, le déplacement au loin est asymptotique à un sinus exponentiellement amorti. Le cas d'une bande finie est aussi étudié mais avec moins de précision. *M. Brelot* (Grenoble).

Differential Equations

***Weise, Karl Heinrich. Gewöhnliche Differentialgleichungen.** Wolfenbütteler Verlagsanstalt, Wolfenbüttel and Hannover, 1948. 146 pp.

A brief introduction to solvable types of differential equations, with examples. Legendre, Bessel, and Sturm-Liouville equations are discussed. Existence theorems as well as solutions by iteration, power series, and numerical methods are mentioned. *P. Franklin* (Cambridge, Mass.).

Mitrinovitch, Dragoslav S. Sur une équation différentielle linéaire du second ordre transformable en elle-même. C. R. Acad. Sci. Paris **228**, 1188–1190 (1949).

Let the function $f(x)$ be representable in the form

$$f(x)=\alpha(\theta'/\theta)+\tfrac{3}{4}(\theta''/\theta)^2-\tfrac{1}{2}\theta'''/\theta',$$

where α is a constant and θ is a function of x. The paper presents transformations on the dependent variables under which the equation $dy/dx+y^2=f(x)$ is transformed into itself. Similar results are also stated for the related equation $d^2z/dx^2-f(x)z=0$. *F. G. Dressel* (Durham, N. C.).

Knebelman, M. S. The Wronskian for linear differential equations. Amer. Math. Monthly **56**, 252–254 (1949).

An explicit computation of the Wronskian of the functions $x^m e^{rx}$ which form a complete set of solutions when the characteristic equation has multiple roots. *P. Franklin.*

González, Mario O. On the solution of ordinary differential equations of the first order invariant under contact transformations. Bull. Amer Math. Soc. **55**, 355–362 (1949).

The paper presents necessary and sufficient conditions for the existence of an infinitesimal contact transformation leaving an ordinary differential equation $u=F(v)$ ($u=u(x, y, p)$, $v=v(x, y, p)$, $p=dy/dx$) invariant. A method is also given of using the infinitesimal contact transformation, when it exists, to solve $u=F(v)$. *F. G. Dressel.*

Popova, N. V. On the transformations produced by the integrals of a class of differential equations. Doklady Akad. Nauk SSSR (N.S.) **65**, 273–274 (1949). (Russian)

The author considers the equation $dw/dt=P(w, t)$, where $P(w, t)$ is a rational function of w for all t in a closed interval $[a, b]$, satisfying the conditions that the number and order of the poles of $P(w, t)$ do not change as t traverses the interval $[a, b]$ and that the poles and residues of $P(w, t)$ at these poles possess continuous derivatives. A number of results concerning the existence, uniqueness and behavior of solutions approaching poles of $P(w, t)$ are stated without proof. The special case where $P(w, t)=w(\mu(t)+w)/(\mu(t)-w)$, $|\mu(t)|=1$, is said to have been discussed by Kučarev.

R. Bellman (Stanford University, Calif.).

***Andronow, A. A., and Chaikin, C. E. Theory of Oscillations. English Language Edition Edited Under the Direction of Solomon Lefschetz.** Princeton University Press, Princeton, N. J., 1949. ix+358 pp. $6.00.

[The Russian original was published in Moscow in 1937.] This text is devoted mainly to the study of nonlinear oscillations. Mathematically the equations considered are the first order system of Poincaré-Bendixon, $\dot{x}=f(x, y)$, $\dot{y}=g(x, y)$ and certain degenerate cases of this as well as cases where the right members contain a parameter. The main interest of the book is for applications. It is very rich in examples from mechanics and electricity and an index of examples is provided at the end. The editor has eliminated some theoretical material from the original Russian text which is not used in the examples and is readily available in other publications. In this way the usefulness of the book to the engineer has been enhanced. *N. Levinson.*

Šestakov, A. A. On the behavior of the integral curves of a system of differential equations in the neighborhood of a singular point of higher order. Doklady Akad. Nauk SSSR (N.S.) **65**, 139–142 (1949). (Russian)

The author considers (1) $\dot{x}_i=X_i=X_1^{(m)}+X_1^{(m+1)}+\cdots$, where the X_i are analytic, $X_1^{(k)}$ is a homogeneous polynomial of degree k, $m\geqq 2$. Let a_i be a simple real solution of the algebraic system $x_1/X_1^{(m)}=x_2/X_2^{(m)}=\cdots=x_n/X_n^{(m)}$. Define the numbers (all functions calculated at a_i):

$$p_{ij}=X_1^{(m)}\cdot(\partial X_{i+1}^{(m)}/\partial x_{j+1})-X_{i+1}^{(m)}\cdot(\partial X_1^{(m)}/\partial x_{j+1}),$$
$$p_{ii}+1=X_1^{(m)}\cdot(\partial X_{i+1}^{(m)}/\partial x_{i+1})-X_{i+1}^{(m)}\cdot(\partial X_1^{(m)}/\partial x_{i+1}),$$
$$p_{in}=X_1^{(m)}X_{i+1}^{(m+1)}-X_{i+1}^{(m)}X_{1+1}^{(m+1)},$$

$i, j=1, 2, \cdots, n-1$, $i\neq j$. The characteristic roots of the matrix (p_{ij}) are called characteristic roots of (1); suppose $\Re(\lambda_i)\neq 0$. Theorem: if $k-1$ characteristic roots have $\Re(\lambda_i)>0$ and $n-k$ have $\Re(\lambda_i)<0$, there exist two integral k-manifolds of (1) passing through 0, one for $x_1>0$, the other for $x_1<0$. The proof is based on results of a previous paper of the author [same Doklady (N.S.) **62**, 591–594 (1948); these Rev. **10**, 251]. *J. L. Massera.*

Šestakov, A. A., and Palvin, A. U. On the asymptotic behavior of solutions of nonlinear systems of differential equations. Doklady Akad. Nauk SSSR (N.S.) **62**, 495–498 (1948). (Russian)

The asymptotic behavior as $t\to\infty$ of the stable solutions of the system

$$dx_i/dt=\sum_{k=1}^{n}a_{ik}x_k+\varphi_i(x_1, x_2, \cdots, x_n, t), \qquad i=1, 2, \cdots, n$$

(a_{ik} constant, $\varphi_i(0, 0, \cdots, 0, t)=0$), has been studied by O. Perron [Math. Z. **29**, 129–160 (1928)] and by I. G. Petrovsky [Rec. Math. [Mat. Sbornik] (1) **41**, 107–155 (1934)]. In the present note some of those results are stated under weaker hypotheses. Not all of the proofs are given.

W. Wasow (Swarthmore, Pa.).

Yakubovič, V. A. On the asymptotic behavior of the solutions of differential equations. Doklady Akad. Nauk SSSR (N.S.) **63**, 363–366 (1948). (Russian)

The system (*) $dx/dt=Ax+f(t, x)$, where x and f are vectors and A is a constant matrix, is considered for the case where A may have multiple characteristic roots and $|f(t, x)|\leqq g(t)|x|$. Using the "variations of constants" formula apparently first applied by Dunkel [Proc. Amer. Acad. Arts Sci. **38**, 339–370 (1902)] to such problems

the author obtains sufficient conditions of the form $\int^{\infty} g(t)t^k dt < \infty$ for solutions of (*) to be asymptotic as $t \to \infty$ to solutions of (*) with $f \equiv 0$. *N. Levinson.*

De Gennaro, Antonio. Alcuni criteri di stabilità per le soluzioni di un'equazione differenziale lineare. Giorn. Mat. Battaglini (4) 2(78), 42–54 (1948).

The author considers the stability and asymptotic behavior of solutions of an nth order linear equation. This is a special case of $dx/dt = P(t)x + Q(t)x + r(t)$, where x and r are vectors and P and Q are square matrices. His case corresponds to the matrix $P(t)$ periodic and $Q(t)$ in some sense small. The author proves various stability criteria not realizing, apparently, that it is a classical result that there exists a periodic nonsingular matrix $Z(t)$ [see, for example, Lefschetz, Lectures on Differential Equations, Princeton University Press, 1946, p. 69; these Rev. 8, 68] such that $dZ/dt + P(t)Z = BZ$, where B is a constant matrix so that the transformation $y = Zx$ yields $dy/dt = By + (ZQZ^{-1})y + Zr$. Since B is constant, this last equation reduces all his results at once to known results for a perturbed system with constant coefficients. *N. Levinson* (Cambridge, Mass.).

Sobol', I. M. On Riccati equations and the reduction to them of linear equations of the second order. Doklady Akad. Nauk SSSR (N.S.) 65, 275–278 (1949). (Russian)

The author is interested in studying the asymptotic behavior and oscillation of the solutions of the linear equation $y'' = a(x)y$. To this end, he utilizes the transformation $y = \exp \int z\,dx$ which transforms the previous equation into the Riccati equation $z' + z^2 = a(x)$. [For a recent application of this method, together with some of the results stated in the paper, cf. Hille, Trans. Amer. Math. Soc. 64, 234–252 (1948); these Rev. 10, 376.] Further results, which seem to be new, concerning the solutions of both the Riccati equation and the linear equation are stated, but no proofs are presented. *R. Bellman* (Stanford University, Calif.).

Gradšteĭn, I. S. On a class of nonlinear differential equations with small coefficients for certain derivatives. Doklady Akad. Nauk SSSR (N.S.) 64, 441–443 (1949). (Russian)

Let $x = (x_1, \cdots, x_p)$, $y = (y_1, \cdots, y_q)$ and consider for $t_0 \leqq t \leqq T$ the real system

$$(1) \qquad \begin{aligned} 0 &= f_i(x, y, t) = \sum g_{ij}(y, t)x_j + g_{i0}(y, t), && i = 1, \cdots, p, \\ dy_i/dt &= f_i(x, y, t), && i = 1, \cdots, q, \end{aligned}$$

and an associated system for $0 < \eta \leqq \eta_0$,

$$(2) \qquad \begin{aligned} dx_i/dt &= F_i(x, y, t, \eta), && i = 1, \cdots, p, \\ dy_i/dt &= F_i(x, y, t, \eta), && i = 1, \cdots, q, \end{aligned}$$

where $F_i = f_i + \eta h_i(x, y, t, \eta)$. The author states a complicated theorem giving a set of sufficient conditions in order that (2) possess a solution tending when $\eta \to 0$ towards a certain preassigned solution of (1). [Relevant references: I. S. Gradšteĭn, C. R. (Doklady) Acad. Sci. URSS (N.S.) 53, 391–394 (1946); same Doklady (N.S.) 59, 841–843 (1948); A. N. Tychonoff, Mat. Sbornik (N.S.) 22(64), 193–204 (1948); G. D. Birkhoff, Trans. Amer. Math. Soc. 9, 219–231 (1908); K. O. Friedrichs and W. R. Wasow, Duke Math. J. 13, 367–381 (1946); Yu-Why Tschen, Compositio Math. 2, 378–401 (1935); these Rev. 8, 328; 9, 437, 588; 8, 272.] *S. Lefschetz* (Princeton, N. J.).

Wendel, J. G. On a van der Pol equation with odd coefficients. J. London Math. Soc. 24, 65–67 (1949).

Generalizing a result of McHarg [same J. 22, 83–85 (1947); these Rev. 9, 435] the author considers

$$(*) \qquad \ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where $f(x)$ and $g(x)$ are odd continuous functions of x and $g(x) > 0$ for $x > 0$. Let $F(x)$ and $G(x)$ be the integrals of $f(x)$ and $g(x)$ and $F(0) = G(0) = 0$. Let (I) $G(\infty) < \infty$, $F(x) \geqq 0$; or (II) $G(\infty) = \infty$ and $F(x) \geqq 0$ for large x. Also let (A) $0 < f(x) < kg(x)$ for some constant k and for small $x > 0$; or (B) $F(x) \leqq 0$ for small $x \geqq 0$. Then if (IA), (IIA), or (IIB) holds all solutions of (*) with initial values $x = 0$, $0 < \dot{x} \leqq (2G(\infty))^{\frac{1}{2}}$ are periodic. [Reviewer's remark: the author has subsequently pointed that in some cases the unicity of the solutions must be set down as an added hypothesis.] *N. Levinson* (Cambridge, Mass.).

Duffin, R. J. Nonlinear networks. III. Bull. Amer. Math. Soc. 55, 119–129 (1949).

[For previous parts cf. same Bull. 52, 833–838 (1946); 53, 963–971 (1947); 54, 119–127 (1948); these Rev. 8, 244; 9, 285, 503.] The author considers the system of differential equations $Ly'' + V(y') + Sy = g$, where y and g are vectors and L and S are symmetric semi-definite matrices and V is a vector function. If $dV = V'(x)dx$, where x is a vector, then $V'(x)$ is a matrix of differential coefficients. It is assumed that $V(0) = 0$, $V'(x)$ is symmetric, and that there exist a symmetric semi-definite matrix R and a constant A such that $A^{-1}(Ry, y) \leqq (V'y, y) \leqq A(Ry, y)$ for any vector y. Here (x, y) is the usual inner product. Using abstract spaces the author establishes that corresponding to a given g there is a unique y. *N. Levinson* (Cambridge, Mass.).

Staržinskiĭ, V. M. On auto-oscillations of an electric governor. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 41–50 (1949). (Russian)

The physical problem is reduced to the mathematical one of determining the possible periodic solutions of the system $J_1\ddot{x} + M_1(\operatorname{sgn} \dot{x}) - k\phi z + M = 0$, $J_2\ddot{y} + M_2(\operatorname{sgn} \dot{y}) - M = 0$, $l\dot{z} + Rz + c_1\dot{x} - c_2(a - y) = 0$, where x, y, z are the time-dependent variables and the other letters represent various physical constants. Necessary and sufficient conditions for the existence of periodic solutions (which are stated without proof) lead to a system of algebraic equations connecting the constants. These equations are solved numerically for different values of the parameters and a table of values is exhibited. *R. Bellman* (Stanford University, Calif.).

Leonov, M. Ya. Certain criteria of dynamical stability. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 737–748 (1948). (Russian)

The author considers the equation

$$G(t)d^2z/dt^2 + (dG/dt + 2v(t))dz/dt + D(t)z = 0,$$

where all functions entering are real. A standard change of independent and dependent variable yields $d^2y/ds^2 + \theta(s)y = 0$. Solutions of the form $y = f(s)\sin\phi(s)$, $dy/ds = u(s)f(s)\cos\phi(s)$ are investigated, and relations are obtained between u, f, and ϕ. After these preliminaries, the case where $\theta(s)$ is periodic with period a is treated. If $\phi(a) = \phi(0) + n\pi$ for some n, then y is said to be a solution with "resonant phase." The conditions for the existence of such a solution are given, together with the connection between this concept and the boundedness or unboundedness of the solution.

R. Bellman (Stanford University, Calif.).

Leonov, M. Ya. The stability of quasi-harmonic oscillations. Doklady Akad. Nauk SSSR (N.S.) **64**, 645–648 (1949). (Russian)

Continuing his investigation of the stability of solutions of the second order linear differential equation with periodic coefficients, the author shows how the equation $\ddot{x}+a(t)\dot{x}+x=0$, $a(t)$ periodic with period $L>0$, $\int_0^L a(t)dt=0$, may be treated using techniques developed in previous papers [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl Mat. Mech.] **10**, 575–580 (1946); same Doklady (N.S.) **62**, 161–162 (1948); these Rev. **8**, 329; **10**, 251, and the paper reviewed above]. *R. Bellman* (Stanford University, Calif.).

Cambi, Enzo. Sulla integrazione delle equazioni differenziali lineari a coefficienti periodici. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 103–123 (1948).

Nach einer allgemeinen Einleitung über die Differentialgleichungen zweiter Ordnung mit periodischen Koeffizienten und ihre Lösungen befasst sich Verf. mit der Ableitung des bekannten Hillschen Lösungsverfahrens. Darauf wendet er sich dem ebenfalls bekannten Verfahren der Kettenbruchentwicklung von Eigenwerten zu. Verallgemeinernd macht er einige Bemerkungen über die Anwendung dieser Verfahren auf lineare Differentialgleichungen $n$ter Ordnung mit periodischen Koeffizienten. *M. J. O. Strutt* (Zürich).

Carafa, Mario. Risoluzione dell'equazione differenziale generale lineare binomia, di ordine qualunque, mediante un numero finito di integrazioni. Mem. Soc. Ital. Sci. (3) **26**, 185–297 (1947).

This is a detailed presentation of results which the author has described briefly in a previous publication [see Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 152–158 (1947); these Rev. **8**, 516]. The chief problem under consideration is that of finding an explicit solution, requiring only a finite number of integrations, of the differential equation $y^{(n)}-a(x)y=f(x)$ and the initial conditions $y(x_0)=y^0, \cdots, y^{(n-1)}(x_0)=y^0_{n-1}$. Here $a(x)$ and $f(x)$ are assumed to be regular analytic functions in a simply connected region in the x-plane. The differential equation and initial conditions are together equivalent to a Volterra integral equation

$$y(x)=\bar{f}(x)+\int_{x_0}^{x}K(x,\xi)y(\xi)d\xi,$$

and it is known that the solution of this can be put in the form

$$y(x)=\bar{f}(x)+\int_{x_0}^{x}R(x,\xi)\bar{f}(\xi)d\xi.$$

However, the usual formulae for calculating the resolvent kernel $R(x,\xi)$ require infinitely many integrations involving the function $a(x)$. The principal result of this paper is that the solution can be written in the form

$$y(x)=\bar{f}(x)+\frac{\partial}{\partial\sigma}\int_{x_0}^{x}d\bar{x}\oint_{\bar{C}'}dz\oint_{C_\mu}d\mu\oint_{C_\omega}d\omega\int_{C_c}\Phi dc\Big|_{\sigma=1},$$

where

$$\Phi=\frac{[M_n]_{k=z-\bar{x}}\bar{f}(\bar{x})}{ca(x)-a(x)\omega^{-1}\displaystyle\int_{x}^{x+\omega(z-\mu)}\Gamma(z-(x-\nu)/\omega)\{a(\nu)\}^{-1}d\nu}.$$

It is unnecessary to explain all of the details of these formulae here, but it is essential to remark that M_n is a function (of x, ξ, z, μ, ω, c, σ) which does not depend upon $a(x)$ or $f(x)$. In principle, therefore, this function can be calculated once for all. Since the determination of $\bar{f}(x)$, in terms of $f(x)$ and the initial conditions, requires one integration, the present formulae give the solution $y(x)$ in terms of seven integrations involving the functions $a(x)$ and $f(x)$.

The reasoning and processes by means of which these conclusions are reached depend heavily upon Fantappiè's theory of analytic functionals, and they are complicated in the extreme. We may describe the procedure vaguely by saying that the author starts with the expression for $R(x,\xi)$ which is given by the usual iterative method for solving the integral equation, and he then transforms this expression through a number of stages until it assumes the desired form. The process involves the consideration of many auxiliary functions and equations. Aside from a few brief remarks concerning the bearing of the results on the theory of the Riccati equation, there is no discussion of possible applications of the theory. *L. A. MacColl.*

Karanikoloff, Christo. Sur une équation différentielle considérée par Kummer. C. R. Acad. Sci. Paris **228**, 888–890 (1949).

For the differential equation $d^ny/dx^n=x^my$ $(m+n\neq 0)$, the author obtains the obvious solutions in powers of x^{m+n}, and discusses the form of these solutions in the exceptional case when $m+n$ is rational with a denominator less than n. *A. Erdélyi* (Pasadena, Calif.).

Gottlieb, Morris J. Oscillation theorems for self-adjoint boundary value problems. Duke Math. J. **15**, 1073–1091 (1948).

This paper is concerned with self-adjoint boundary value problems involving the second order differential equation $(R(x,\lambda)y')'+P(x,\lambda)y=0$, $a\leqq x\leqq b$, and boundary conditions in the end-values of $y(x)$ and $R(x,\lambda)y'(x)$ at $x=a$ and $x=b$. Using the methods of Morse [The Calculus of Variations in the Large, Amer. Math. Soc. Colloquium Publ., v. 18, New York, 1934, chapter IV], the author presents a systematic classification of such boundary value problems and establishes oscillation theorems for the various classes of problems. *W. T. Reid* (Evanston, Ill.).

Macfarlane, G. G. The application of a variational method to the calculation of radio wave propagation curves for an arbitrary refractive index profile in the atmosphere. Proc. Phys. Soc. **61**, 48–59 (1948).

The basic problem is to find the eigenvalues D_m and eigenfunctions U_m of the wave equation

$$d^2U_m/ds^2+\{s+f(s)+D_m\}U_m=0,$$

in which $f(s)$, the M-anomaly, is a given function, which tends to zero as s tends to infinity. In this paper a method, previously described by the author [Proc. Cambridge Philos. Soc. **43**, 213–219 (1947); these Rev. **8**, 550] is applied to determine the first eigenvalue D_1 for the class of M-anomalies that can be represented by a series of the form $\sum_{n=1}^{r}A_n\exp(-\alpha_n s)$. Two numerical examples are given to illustrate the method. *From the author's summary.*

Titchmarsh, E. C. On the uniqueness of the Green's function associated with a second-order differential equation. Canadian J. Math. **1**, 191–198 (1949).

Let $G(x,\xi,\lambda)$ be the Green's function for the differential equation

(*) $$d^2\varphi/dx^2+\{\lambda-q(x)\}\varphi=0$$

on the interval $(0, \infty)$. It has the following properties. (1) If $x \neq \xi$, $G(x, \xi, \lambda)$ has continuous partial derivatives with respect to x up to the second order, and satisfies (*) as a function of x; (2) for some real value of α,

$$G(0, \xi, \lambda) \cos \alpha + G_x(0, \xi, \lambda) \sin \alpha = 0;$$

(3) G belongs to $L^2(0, \infty)$ as a function of x; (4) $G(x, \xi, \lambda)$ is continuous at $x = \xi$, while $G_x(\xi+, \xi, \lambda) - G_x(\xi-, \xi, \lambda) = -1$. It is proved that if $q(x) \geqq -Ax^2 - B$, where A and B are nonnegative constants, then the Green's function is unique. A similar result is established for the equation

$$\nabla^2 \varphi + \{\lambda q(x, y)\} \varphi = 0,$$

the region being the whole plane. *H. Pollard.*

Hylleraas, Egil A. Calculation of a perturbing central field of force from the elastic scattering phase shift. Physical Rev. (2) **74**, 48–51 (1948).

Let u_1, u_2; v_1, v_2 be the solutions of the equations

$$\{L + U(r)\} u = 0, \quad \{L + V(r)\} v = 0;$$
$$L \equiv d^2/dr^2 - l(l+1)/r^2 + k^2,$$

u_1 and v_1 vanishing at $r = 0$. The potentials $U(r)$ and $V(r)$ are supposed to be such that asymptotic phases $\xi(k)$ and $\eta(k)$ can be defined for u_1 and v_1, e.g.,

$$u_1 \to \sin(kr - \tfrac{1}{2}l\pi + \xi(k)),$$

$r \to \infty$ (Coulomb forces thus excluded). The singular solutions u_2 and v_2 are chosen so that, for instance, $u_2 \to \cos(kr - \tfrac{1}{2}l\pi + \xi(k))$, $r \to \infty$.

The following identity is known [L. Hulthén, Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **14**, no. 8 (1944); these Rev. **6**, 110]:

$$\text{(I)} \qquad k \sin(\eta - \xi) = \int_0^\infty (V - U) u_1 v_1 dr.$$

The main point of the present paper is to establish the inversion of (I); the author finds

$$\text{(II)} \qquad V - U = -\frac{4}{\pi} \int_0^\infty \sin(\eta - \xi) \frac{d}{dr}(u_1 v_2 + u_2 v_1) dk,$$

which is a generalization of a result due to Fröberg [Ark. Mat. Astr. Fys. **34A**, no. 28 (1948); these Rev. **10**, 120]. If the phase shift η is given as a function of k the corresponding potential V can be obtained from (II) by means of successive approximations: substituting the nth approximation $V^{(n)}$ for U and u_1, u_2 for v_1, v_2, respectively, we get $V^{(n+1)}$ from (II). [This is virtually the method of Fröberg, loc. cit.: reviewer's remark.] The convergence of the method is illustrated by a simple example, $V = -c/r^2$, which is on the very limit of applicability. *L. Hulthén* (Stockholm).

Glazman, I. On the deficiency index of differential operators. Doklady Akad. Nauk SSSR (N.S.) **64**, 151–154 (1949). (Russian)

In attempting to generalize a well-known result of Weyl [Math. Ann. **68**, 220–269 (1909)], Windau [Math. Ann. **83**, 256–279 (1921)], for $n = 2$, and Schin [Rec. Math. [Mat. Sbornik] N.S. **7(49)**, 479–532 (1940); these Rev. **2**, 51], for general n, make the following statements concerning the number of linearly independent solutions in $L_2(0, \infty)$ of the ordinary self-adjoint differential equation of order $2n$, (*) $l(y) = \lambda y$, for nonreal λ: (A) the number m of such solutions is not less than n; (B) either $m = n$ or $m = 2n$. By consideration of the deficiency index of a closed Hermitian operator in Hilbert space the author proves (A) and disproves (B). [For the formulation of the problem in terms of Hilbert space see Kreĭn, Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 431–495 (1947); Mat. Sbornik N.S. **21(63)**, 365–404 (1947); these Rev. **9**, 515, where the obvious modifications have to be made.] The minimal quasi-differential operator L belonging to l is defined as the closure of operators which map φ on $l(\varphi)$, where φ satisfies the conditions: (i) $\varphi(t) = 0$, for $t > a_\varphi$ $(0 \leqq a_\varphi < \infty)$, (ii) $\varphi^{(k)}(t)$ is absolutely continuous, (iii) $\varphi^{(k)}(0) = 0$ $(k = 0, \cdots, 2n-1)$. Since the domain of the adjoint L^* consists of the functions $\psi(t)$ satisfying the conditions: $\psi^{(k)}(t)$ absolutely continuous, $\psi(t), l(\psi) \in L_2(0, \infty)$, and $L^*\psi = l(\psi)$, the statement (A) above is equivalent to the following theorem. The deficiency index (m, m) of L satisfies the inequality $n \leqq m \leqq 2n$.

The counter-example used to disprove (B) is based on the differential operator $l = l_1^{n-s} l_2^{2s} l_1^{n-s}$, where

$$l_1 = -i(1+t)D(1+t), \quad l_2 = D^2 - 1,$$

$n \leqq m \leqq 2n$. The minimum quasi-differential operator L is shown to have deficiency index (m, m). This is done by virtue of the statement that if L_1, L_2, L are the operators belonging to l_1, l_2, and $l_1 l_2 l_1$ respectively, then def $L \geqq$ def $L_1 L_2 L_1$, and the observation that $l(y) = 0$ has $2n - m$ linearly independent solutions. *M. J. Gottlieb* (Chicago, Ill.).

Terracini, Alejandro. Geometric characterizations of the equation (G) subordinate to a differential equation of type (F). Univ. Nac. Tucumán. Revista A. **6**, 255–261 (1948). (Spanish)

By a differential equation of type (F) is meant an equation of the form $y''' = F(x, y, y') + G(x, y, y')y'' + H(x, y, y')y''^2$. In particular, if $F \equiv 0$, the equation is said to be of type (G). If an equation of type (F) and an equation of type (G) involve the same functions G and H, the second equation is said to be subordinate to the first. The author has previously given geometrical characterizations of the families of curves defined by equations of types (F) and (G), interpreting the variables x, y as nonhomogeneous Cartesian coordinates in a projective plane [same Revista **2**, 245–329 (1941); **3**, 195–234 (1942); these Rev. **4**, 54, 256]. In the present paper he gives a characterization of the family of curves defined by an equation of type (G) that is subordinate to an equation of type (F) in terms of the characterization of the family of curves defined by the latter equation. In the latter part of the paper, he gives a new geometrical discussion of these equations, in which the variables x, y are interpreted as certain coordinates upon a surface of Veronese in a five-dimensional projective space. *L. A. MacColl* (New York, N. Y.).

Terracini, Alejandro. On the differential equations of type (G) and of type (F). Univ. Nac. Tucumán. Revista A. **6**, 273–287 (1948). (Spanish)

The author here gives the following new characterizations of the families of curves defined by differential equations of the types (F) and (G) described in the preceding review. Consider the family of curves, in the (x, y)-plane, defined by a general third order differential equation $y''' = f(x, y, y', y'')$. If $P_1 : (x_1, y_1)$ and $P_2 : (x_2, y_2)$ are any two neighboring points, there are ∞^1 curves of the family which pass through both points. Consider the pencils of tangents to these curves at P_1 and P_2. The fact that the correspondence between the pencils, in which the two tangents to the same curve correspond, is a projectivity would be expressed by a certain equation $\Phi = 0$. Here Φ is a function of $x_2 - x_1$, $y_2 - y_1$, the

slopes m_1 and m_2 of the tangents to the typical curve at P_1 and P_2, and of the derivatives dm_2/dm_1 and d^2m_2/dm_1^2. In general, the equation is not satisfied identically, the expansion of Φ in terms of x_2-x_1 beginning with a term in $(x_2-x_1)^{11}$. If this term vanishes, so also does the term in $(x_2-x_1)^{12}$, and the expansion usually begins with a term in $(x_2-x_1)^{13}$. In this case the author says that the family of curves has the projective property to the first approximation. He shows that if a family of curves has the projective property to the first approximation, the defining differential equation is of type (F), and conversely. We may proceed similarly, considering perspectivities instead of projectivities. It is found that in order that a family of curves shall have the perspective property to the first approximation (in a certain slightly altered sense) it is necessary and sufficient that the defining differential equation be of the type (G).

L. A. MacColl (New York, N. Y.).

Cinquini-Cibrario, Maria. Sopra il problema di Cauchy per i sistemi di equazioni alle derivate parziali del primo ordine. Rend. Sem. Mat. Univ. Padova **17**, 75–96 (1948).

The author considers the system of n equations

$$F_i(x, y, z_1, \cdots, z_n, p_1, \cdots, p_n, q_1, \cdots, q_n)=0$$

for n functions $z_i(x, y)$ and their first partial derivatives p_i and q_i. In the notation of the author the F_i are assumed to be of class D^{III}-L in their arguments, i.e., they shall have third derivatives satisfying a Lipschitz condition. The system shall be hyperbolic in the sense that the characteristic equation has n distinct real roots. The Cauchy problem to be solved consists in finding a system of solutions $z_i(x, y)$ for which along a curve $y=f(x)$ the z_i, p_i, q_i are prescribed, so as to satisfy the obvious compatibility relations. The Cauchy data f, z_i, p_i, q_i shall be of class D^{II}-L in x, and constitute a noncharacteristic strip. It is proved that under these assumptions there exists one and only one solution of the Cauchy problem of class D^{III}-L in a sufficiently small neighbourhood.

The author proves this theorem by reducing the problem to a canonical type, for which the existence and uniqueness of solutions was proved in a previous paper [Ann. Mat. Pura Appl. (4) **24**, 157–175 (1945); these Rev. **9**, 441]. This reduction is achieved by referring the variables x, y, z_i, p_i, q_i corresponding to an integral surface to parameters λ, μ, such that the curves $\lambda=$constant and $\mu=$constant are two of the n families of characteristic curves on the surface, and the initial curve corresponds to $\lambda+\mu=0$.

F. John (New York, N. Y.).

Ingersoll, Benham M. The regularity domains of solutions of linear partial differential equations in terms of the series development of the solution. Duke Math. J. **15**, 1045–1056 (1948).

The equation considered is of the form

$$\Delta u+Au_x+Bu_y+Cu=0, \tag{1}$$

where A, B, C are real-valued entire functions of x and y. Setting $x+iy=z$, $x-iy=z^*$, it may be written in the form (2) $U_{z^*}+au_z+a^*u_{z^*}+cu=0$. It is shown that under certain conditions the knowledge of the coefficients of $z^m z^{*n}$ (n fixed) in the power series development of a complex-valued solution $u(z, z^*)$ of (2) permits one to find a domain in which $u(z, z^*)$ is regular. The coefficients a, c of (2) do not enter in this determination. In the case of a real-valued solution $u(x, y)=U(z, z^*)$ of (1) the knowledge of the coefficients of $z^m z^{*n}$ (or of $z^m z^{*n}$, n fixed) in the Taylor expansion of U, together with the knowledge of some subsequences of the coefficients of the Taylor expansions of A, B, C, determines the regularity domain of u. Some extensions are indicated. The author bases his investigation on Bergman's operator method [Trans. Amer. Math. Soc. **53**, 130–155 (1943); these Rev. **4**, 159].

L. Bers (Princeton, N. J.).

Herzfeld, Karl F. Betrachtungen über Knotenflächen bei Schwingungsproblemen. Z. Naturforschung **3a**, 457–460 (1948).

This article contains some generalizations of Sommerfeld's results concerning the nodal lines and surfaces of the eigenvalue problem $\Delta_n\Phi+F\Phi=0$, with suitable boundary conditions (F contains a parameter). The author's summary states: "Das Vorzeichen der Amplitude oder Wellenfunktion ist auf den zwei Seiten einer Knotenfläche oder -linie verschieden. Daher muss eine solche Knotenfläche entweder geschlossen sein, oder ins Unendliche gehen, oder am Rande enden. Bei einer adiabatischen Änderung der Schwingung kann sich die Zahl der Knotenflächen ändern."

M. J. Gottlieb (Chicago, Ill.).

Ghizzetti, Aldo. Applicazione del metodo della trasformata parziale di Laplace al problema di Dirichlet per l'equazione $\Delta_n u-\lambda^2 u=F$ in n variabili. Rend. Sem. Mat. Univ. Padova **17**, 39–74 (1948).

The problem of Dirichlet for the partial differential equation (1) $\nabla^2 u-\lambda^2 u=F$ (λ real) for a region in n-dimensional space bounded by a "tubular" hypersurface Γ and the planes $x=0$ and $x=1$ is reduced to an infinite system of integral equations for the unknown normal derivative of u on Γ. The author has previously discussed the two-dimensional case [Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 1–47 (1947); these Rev. **9**, 94]. The idea is well illustrated by the case $n=3$. The surface Γ is such that any plane $x=\xi$ cuts it in a simple closed curve γ_ξ; the area in $x=\xi$ bounded by γ_ξ is denoted by Γ_ξ. Then the partial Laplace transform of $u(x, y, z)$, defined by

$$u^*(x; p_1, p_2)=\int\int_{\Gamma_x} u(x, y, z)\exp(p_1y+p_2z)dydz,$$

satisfies

$$d^2u^*/dx^2+(p^2-\lambda^2)u^*=V(x; p_1, p_2),$$

where $p^2=p_1^2+p_2^2$. The terminal values

$$a_0^*(p_1, p_2)=u^*(0; p_1, p_2),\quad a_1^*(p_1, p_2)=u^*(1; p_1, p_2)$$

are known functions, but V depends, not only on the known values of u on Γ, but also on the unknown values of its normal derivative.

Writing $\bar\omega^2=p^2-\lambda^2$, we have

$$u^*(x; p_1, p_2)=\frac{\sin\bar\omega(1-x)}{\sin\bar\omega}\Big[a_0^*(p_1, p_2)-\int_0^x\frac{\sin\bar\omega\xi}{\bar\omega}V(\xi; p_1, p_2)d\xi\Big]+\frac{\sin\bar\omega x}{\sin\bar\omega}\Big[a_1^*(p_1, p_2)-\int_x^1\frac{\sin\bar\omega(1-\xi)}{\bar\omega}V(\xi; p_1, p_2)d\xi\Big].$$

But since u^* is an integral function of p, the singularities at $\bar\omega=n\pi$ cannot occur. This requires that, when

$p_1^2+p_2^2=n^2\pi^2+\lambda^2$,

$$\sin n\pi(1-x)\left[a_0^*(p_1,p_2)-\int_0^x \frac{\sin n\pi\xi}{n\pi}V(\xi;p_1,p_2)d\xi\right]$$
$$+\sin n\pi x\left[a_1^*(p_1,p_2)-\int_x^1 \frac{\sin n\pi(1-\xi)}{n\pi}V(\xi;p_1,p_2)d\xi\right]=0,$$

which reduces to

$$\int_0^1 \sin n\pi\xi\cdot V(\xi;p_1,p_2)d\xi=n\pi[a_0^*(p_1,p_2)-(-1)^n a_1^*(p_1,p_2)].$$

As V involves the unknown normal derivative of u on Γ, this is an infinite system of integral equations for the unknown function.

After a discussion of the Green's function for the differential equation (1) in $0\leqq x\leqq 1$, the author proves rigorously the equivalence of the problem of Dirichlet and the system of integral equations.

E. T. Copson (Dundee).

Brelot, M. Sur le principe des singularités positives et la notion de source pour l'équation (1) $\Delta u(M)=c(M)u(M)$, $(c\geqq 0)$. Ann. Univ. Lyon. Sect. A. (3) **11**, 9–19 (1948).

The author extends some of his earlier studies [Bull. Sci. Math. **60**, 112–128 (1936)] on the (heat) equation in the neighborhood of an isolated singularity O, which plays the role of a "simple point source." In (1) above, $c(M)$ is finite, nonnegative, and continuous in a deleted neighborhood of the isolated singularity O, and $\Delta u(M)$ is a suitably chosen generalized Laplacian of the continuous integral $u(M)$ of (1). The results depend upon a careful study of what may be called "generalized flux," and include the following result. The author exhibits two integrals of (1), $u_1(M)$ and $u_2(M)$, for the two-dimensional case, such that u_1 and u_2 both vanish on a small contour γ enclosing O, and $u_1>0$ and $u_2>0$ inside γ, and such that $u_1(M)$ and $u_2(M)$ are not proportional.

M. Reade (Ann Arbor, Mich.).

Eichler, Martin. Allgemeine Integration linearer partieller Differentialgleichungen von elliptischem Typ bei zwei Grundvariablen. Abh. Math. Sem. Univ. Hamburg **15**, 179–210 (1947).

The author considers partial differential equations of the second order of the form (1) $\Delta u+Au_x+Bu_y+Cu=0$ in the variables x, y, where A, B, C are real analytic functions of x, y. For a special form of (1), he obtains the solution $u(x,y)=f(z)-\int_0^z S(x,y,\zeta)f(\zeta)d\zeta=\mathfrak{S}[f(z)]$ to be considered as an integral operator on the arbitrary analytic function $f(z)$, $z=x+iy$. Here S is assumed to satisfy the following conditions: (1) S is a solution of the differential equation, (2) S is an analytic function of ζ, (3) $T(x,y)=S(x,y,x+iy)$ satisfies the differential equation $T_x+iT_y=\frac{1}{2}N$, where N is a coefficient of the special form of (1) considered. The properties of the operator $\mathfrak{S}$ are investigated in some detail and applications of it are made to particular partial differential equations.

However, in a series of papers prior to the author's, Bergman investigated such integral operator solutions so that essentially all the work of this paper can be found in the work of Bergman. In an added note the author points out that the earlier work of Bergman came to his attention only after his paper was in print.

A. Gelbart (Syracuse, N. Y.).

Eichler, Martin M. E. On the differential equation $u_{xx}+u_{yy}+N(x)u=0$. Trans. Amer. Math. Soc. **65**, 259–278 (1949).

This paper is a sequel to the author's earlier paper [see the preceding review]. Using the general results of the previous paper, which are duplications of Bergman's earlier work, the author investigates properties of the solutions of the differential equation (1) $\Delta u+N(x)u=0$, $u=u(x,y)$, where $N(x)$ is a function representable in the form

$$N(x)=x^{-\rho}(c_0+c_1x^\rho+c_2x^{2\rho}+\cdots),$$

$\rho>0$. The integral operator $u(x,y)=f(z)-\int_{z_0}^z S(x,y,\zeta)f(\zeta)d\zeta$, where $f(z)$ is an arbitrary analytic function of z, $z=x+iy$, is a solution of (1). Here S is to satisfy certain restrictive conditions. The integral operator can be put in the form

$$(2)\quad u(x,y)=f(z)-p_1(x)\int_{z_0}^z f(\zeta)d\zeta$$
$$+P_2(x)\int_{z_0}^z\int_{z_0}^\zeta f(\eta)d\eta d\zeta+\cdots$$

or (3) $u(x,y)=q_0(x)f(z)+q_1(x)f'(z)+q_2(x)f''(z)+\cdots$, where p_i and q_i are assumed to satisfy certain special conditions. The author investigates certain "canonical" forms of S, the convergence of series (2) and (3) in the neighborhood of $x=0$, where $N(x)$ is assumed to be singular, and the connection between the two series. As in some of Bergman's work, extensions to differential equations of more than two variables are made. *A. Gelbart* (Syracuse, N. Y.).

Bergman, Stefan. An initial value problem for a class of equations of mixed type. Bull. Amer. Math. Soc. **55**, 165–174 (1949).

The equations considered are (1a) $\psi_{xx}+(-x)^s\psi_{yy}=0$ $(x<0)$, (1b) $\psi_{xx}-x^s\psi_{yy}=0$ $(x>0)$, where s is a constant, $s>-1$. It is required to find a solution of (1) defined for sufficiently small (a, negative, b, positive) values of x, which satisfies the initial conditions $\psi(0,y)=T(y)$, $\psi_x(0,y)=S(y)$. The author assumes that the functions $T(y)$, $S(y)$ are analytic and obtains explicit formulae for the solution, which involve complex integration and the values of T and S for complex y. He also discusses properties of the solution in the case when T and S are entire functions. *L. Bers.*

Graffi, Dario. Il teorema di unicità nella teoria della propagazione del calore in un mezzo di conduttività variabile con la temperatura. Atti Sem. Mat. Fis. Univ. Modena **2**, 143–148 (1948).

Let $\varphi(u)$ be a monotone increasing function, and $D(u)$ a positive function of class C'. Denote by R a region, and by $\sigma=\sigma_1+\sigma_2$ the boundary of R. The author shows the following boundary value problem has at most one solution. In R the function $u(x,y,z;t)$ satisfies the equation $u_{xx}+u_{yy}+u_{zz}=D(u)u_t$, for $t>0$. Initially $u(x,y,z,0)$ is assigned in R. On σ_1 the value of u is assigned for all t, while on σ_2 the derivative of u taken in the direction of the interior normal is equal to $\varphi(u)$. *F. G. Dressel* (Durham, N. C.).

Bedini, Lidia. Integrazione di una particolare equazione a derivate parziali. Atti Sem. Mat. Fis. Univ. Modena **2**, 176–196 (1948).

A solution and also an approximate solution are given for the following boundary value problem. For $0<x<l$, $0<t$ the function $u(x,t)$ is a solution of (*) $v^2(t)u_{xx}=u_{tt}+b(t)u_t+c(t)u$, boundary and initial conditions on u being $u(0,t)=u(l,t)=0$, $u(x,0)=f(x)$, $u_t(x,0)=g(x)$. The author extends the sepa-

ration of variables methods used by Paolini [Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 4, 1–24 (1943); these Rev. 8, 332] on equation (*) with $b=c=0$.

F. G. Dressel (Durham, N. C.).

Gårding, Lars. Une propriété caractéristique des équations hyperboliques à coefficients constants. C. R. Acad. Sci. Paris 228, 731–732 (1949).

The result given in this note connects hyperbolic character in a certain "algebraic" sense with a type of "continuous dependence" of solutions on data in a hyperplane. Let $q(\partial/\partial x)f=0$ be a partial differential equation for a complex-valued function f of the real variables $x_1, \cdots, x_n$. Let q be a polynomial with constant complex coefficients of the form $p+r$, where p is the homogeneous polynomial formed by the highest order terms of q. Let $\mathfrak{A}(q)$ be the manifold of solutions f which are infinitely often differentiable in the whole space. A real direction ξ is called hyperbolic if $p(\xi)\neq 0$ and if there exists a t_0 such that $q(t\xi+i\eta)\neq 0$ for all real η, if $t>t_0$. The author states that this definition of hyperbolic character is equivalent to the following one: for every sequence of functions $f_1, f_2, \cdots$ belonging to $\mathfrak{A}(q)$, which is such that the f_k and each of their derivatives converge to 0 uniformly on every compact subset of the hyperplane $y\cdot\xi=0$, the f_k and each of their derivatives converge to 0 uniformly on every compact subset of the whole space.

F. John (New York, N. Y.).

Castelluccio, Domenico. Il metodo dell'onda di spessore infinitesimo per l'analisi dei fenomeni di propagazione per onde. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 10(79), 273–307 (1 plate) (1946).

A one-dimensional wave phenomenon is thought of as composed of two progressive wave components traveling in opposite directions. The author describes how a wave motion can be determined from initial conditions if, for each of the two components the velocity of propagation, the coefficient of reflection and the coefficient of absorption are known as functions of position and time, provided the principle of superposition holds.

F. John.

***Deprima, Charles R. Uniqueness Theory for Linear Hyperbolic Partial Differential Equations.** Abridgment of a thesis, New York University, 1948. ii+15 pp.

The author studies hyperbolic systems of first order linear differential equations. The uniqueness of the solution of the initial value problem is proved under certain conditions of symmetry and regularity on the coefficients. The theory of matrices combined with a generalization of the energy method is used in the proof. It is also shown that the theory developed applies to second order linear hyperbolic equations. The case where the coefficients of the equations have singularities on the initial surface is also discussed.

P. A. Lagerstrom (Pasadena, Calif.).

Élianu, Jean. Les opérateurs différentiels et la construction de la solution élémentaire des équations aux dérivées partielles polyhyperboliques dans le cas impair. C. R. Acad. Sci. Paris 228, 1095–1096 (1949).

Élianu, Jean. Les opérateurs différentiels et la construction de la solution élémentaire des équations aux dérivées partielles polyhyperboliques dans le cas pair. C. R. Acad. Sci. Paris 228, 1186–1188 (1949).

The differential equations considered are of the form $F^p u=0$, where F is a second order linear nonparabolic differential operator with analytic coefficients, and F^p is obtained by p-fold iteration of F. The author outlines the construction of an "elementary" solution for equations of this type, generalizing the results of Hadamard for the case $p=1$.

F. John (New York, N. Y.).

Mangeron, D. I. Sur quelques propriétés des valeurs caractéristiques des spectres concernant certains problèmes aux limites. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 2, 372–374 (1947).

Allowing for apparent typographical errors, it seems to the reviewer that the author announces a formula for the variation of the mth eigenvalue of the following Dirichlet-type problem for the domain D $(x_i'\leqq x_i\leqq x_i'')$, $i=1, \cdots, n$:

$$\frac{\partial^{2n}u}{\partial x_1^2 \cdots \partial x_n^2}-\lambda A(x_1, \cdots, x_n)u=0,$$

$u=0$, on the boundary of D. He states that it is possible to obtain similar results for a class of partial differential equations which are reducible in the sense of Picone [Atti del secondo Congresso dell'Unione Matematica Italiana, 1940, pp. 230–238; these Rev. 3, 122]. The formula seems to be incorrect.

M. J. Gottlieb.

Difference Equations, Special Functional Equations

De Bruijn, N. G. The asymptotically periodic behavior of the solutions of some linear functional equations. Amer. J. Math. 71, 313–330 (1949).

The linear functional equations considered are of the type $L(f)=w(x)f'(x)+p(x)f(x)-q(x)f(x-1)=r(x)$, $x\geqq 1$, $f(x)=f_0(x)$, $0\leqq x\leqq 1$. A subtype whose coefficients satisfy the conditions imposed by the author is represented by the equation $x^{-a}f'(x)+f(x)-f(x-1)=0$. It is shown that, if $a>1$, $f(x)$ tends to a periodic function as $x\to\infty$. If $\frac{1}{2}<a<1$, the behavior is quite different and more complicated, while for $0<a\leqq\frac{1}{2}$, the asymptotic behavior is not known. The author states that, in the case $a\leqq 0$, every solution tends to a constant. A detailed investigation of the operator is required to prove the stated results, and, in the course of this, completeness and uniqueness theorems are obtained. A consequence is that for $a>1$, $f\to 0$ as $x\to\infty$ implies $f\equiv 0$. The foregoing results are then applied to some equations of the type $g'(x)=k(x)g(x-1)$, $k(x)$ positive and continuous for $x\geqq 1$, and, in particular, to a functional equation, $F'(y)=F(y/r)$, $r>1$ (set $y=r^x$, $F(r^x)=G(x)$), derived by Mahler in connection with a partition problem in the theory of numbers [J. London Math. Soc. 15, 115–123 (1940); these Rev. 2, 133].

R. Bellman.

Mambriani, A. Sulla risoluzione di una equazione funzionale con l'algebra delle successioni. Atti Sem. Mat. Fis. Univ. Modena 1, 143–147 (1947).

The author expresses, in terms of his "algebra of sequences" [Ann. Mat. Pura Appl. (4) 8, 103–139 (1930); 9, 25–56 (1931)], the polynomial solutions of the linear differential-difference equation with constant coefficients $\sum_{\mu=0}^{p}\sum_{\nu=0}^{q}h_{\mu,\nu}\varphi^{(\mu)}(x+\omega_\nu)=x^n$.

W. Strodt.

Robinson, Lewis Bayard. A rigorous proof that a given function is finite valued. Revista Ci., Lima 50, 141–150 (1948).

The author discusses the boundedness, along infinitely many radii of the unit circle, of solutions $u(x)$ of certain func-

tional equations, among them the equation

$$u'(x)=\lambda(1-x^2)^{-1}u(x^2).$$

I. M. Sheffer (State College, Pa.).

Ionesco, D. V. Sur une équation de récurrence à deux indices. Disquisit. Math. Phys. **6**, 49–80 (1948).

The Stirling numbers of the second kind, $S(k,n)=\Delta^k 0^n/k!$, are first generalized to

$$A(k+1,n+1)=\Delta^k(1+u+\lambda)(1+u+2\lambda)\cdots(1+u+n\lambda)/k!,$$

which have the recurrence

$$A(k,n+1)=(u+n\lambda+k)A(k,n)+A(k-1,n).$$

This in turn is generalized to

$$B(k,n+1)=(u+n\lambda+k\sigma)B(k,n)+B(k-1,n)$$

and solutions are given for six sets of boundary conditions, chosen so as to permit arbitrary assignment of the numbers on the column $(0,n)$ and the diagonal (k,k), or column $(0,n)$ and row $(n,0)$. *J. Riordan* (New York, N. Y.).

Simaika, J. B. On a difference equation in renewal theory. Proc. Math. Phys. Soc. Egypt **3** (1947), 15–23 (1948).

The renewal equation $u_n=\sum b_k u_{n-k}$ is treated by the method of generating functions. It is assumed that $\sum b_k x^k$ is a rational function and the method of partial fractions is applied. The results are then generalized to the case $u_n+f_n=\sum b_k u_{n-k}$, where f_n is given. *W. Feller.*

Montel, Paul. Sur un système d'équations aux différences finies. C. R. Acad. Sci. Paris **227**, 877–879 (1948).

L'auteur considère les différences successives d'une fonction continue $f(x,y)$:

$$\Delta_h[f]=f(x+\eta,y+\eta')-f(x,y),\qquad h=\eta+i\eta';$$
$$\Delta_{h_1h_2}[f]=\Delta_{h_1}[\Delta_{h_2}[f]]=\Delta_{h_2}[\Delta_{h_1}[f]],$$
$$\Delta_{h_1h_2\dots h_n}[f]=\Delta_{h_1}[\Delta_{h_2h_3\dots h_n}[f]],$$

et démontre: si $f(x,y)$ vérifie 3 équations du type

$$\Delta_{h_1h_2\dots h_n}[f]=0,$$

les nombres h_j, k_r, l_s $(j,r,s=1,2,\cdots,n)$ étant indépendants, cette fonction est un polynome de degré au plus $2(n-1)$. Pour que le polynome homogène $f(x,y)$ vérifie une équation du type précédent, il faut et il suffit qu'il soit de la forme $\sum_1^n a_i(x-\alpha_i y)^m$, où les a_i sont des coefficients constants et les α_i sont les quotients η_i/η_i'. Si une fonction analytique $f(x,y)$ des variables complexes x,y vérifie 5 équations du type indiqué, obtenues en posant

$$\Delta_h[f]=f(x+\eta,y+\eta')-f(x,y)$$

avec η,η' complexes, alors, si les accroissements h_j,k_r,l_s,u_t,v_w $(j,r,s,t,w=1,2,\cdots,n)$ sont indépendants, $f(x,y)$ est un polynome de degré au plus $2(n-1)$. Même résultat si le module construit sur les accroissements est dense sur une variété linéaire à 3 dimensions réelles. *P. Lelong.*

Montel, Paul. Sur quelques équations aux différences mêlées. Ann. Sci. École Norm. Sup. (3) **65**, 337–353 (1948).

Let $h_r=h_r'+ih_r''$, $k_r=k_r'+ik_r''$, $l_r=l_r'+il_r''$ be complex numbers and let $\Delta_{h_r}[f]$ denote the difference

$$\Delta_{h_r}[f]=f(x+h_r',y+h_r'')-f(x,y),$$

where $f(x,y)$ is a function of the two real variables x,y. Let three complex numbers h,k,l be called independent if there are no three integers m,n,p such that $m\eta_1+n\eta_2+p\eta_3=0$, where the three real numbers η_1,η_2,η_3 are determined (up to a common factor) by the relation $\eta_1 h+\eta_2 k+\eta_3 l=0$. The author studies the possible solutions of the system of difference equations $\Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_n}[f]=0$, $\Delta_{k_1}\cdots\Delta_{k_n}[f]=0$, $\Delta_{l_1}\cdots\Delta_{l_n}[f]=0$, where all the triples (h_i,k_j,l_s) $(i,j,s=1,\cdots,n)$ are independent. It is found that all possible solutions of this system are polynomials. It is further shown that, in addition to the arbitrary homogeneous polynomials of degree $N\leqq n-1$ which trivially satisfy the system of equations, there exist nontrivial homogeneous polynomial solutions of degree $n\leqq N\leqq 2(n-1)$. Similar results are shown to hold in the case in which both the variables x, y and the increments h_r', h_r'', etc., are complex. *Z. Nehari* (St. Louis, Mo.).

Integral Equations

***Hamel, Georg. Integralgleichungen. Einführung in Lehre und Gebrauch.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1949. viii+166 pp. 15.60 DM.

"Zweite, berichtigte Auflage." The first edition appeared in 1937.

Yagi, Fumio. On certain Stieltjes integral equations. Univ. Washington Publ. Math. **3**, no. 1, 21–30 (1948).

This paper deals with the integral equation

$$(1)\qquad g(z)=\int_{-\infty}^{\infty} f(z-\lambda)dk(\lambda),$$

where the (complex-valued) kernel $k(\lambda)$ is a nonsingular function of bounded variation on every finite interval of the real variable λ. It is assumed that the Fourier-Stieltjes transform

$$K(\omega)=K(s+it)=\int_{-\infty}^{\infty} e^{\omega\lambda}dk(\lambda)$$

is absolutely convergent in the strip $I:\alpha\leqq s\leqq\beta$ and has a finite number N of zeros of order μ_n at $\omega_n=s_n+it_n$ in I and is bounded away from zero in I except in the neighborhood of $\omega_1,\cdots,\omega_N$. It is further assumed that $g(z)=g(x+iy)$ is analytic in the strip $J:a<y<b$ and that $N(g;\alpha,\beta)<\infty$, where

$$N(g;\alpha,\beta)=\sup_{\gamma\in J}\left\{\int_{-\infty}^{0} e^{\beta x}|g(z)|\,dx+\int_{0}^{\infty} e^{\alpha x}|g(z)|\,dx\right\}.$$

Under these conditions it is shown that (1) has a solution $f(z)$ analytic in J for which $N(f;\alpha,\beta)<\infty$. Moreover, the set of all solutions $\bar f(z)$ of (1) satisfying $N(\bar f;\alpha,\beta)<\infty$ is given by

$$\bar f(z)=f(z)+\sum_{n=1}^{N}\sum_{m=1}^{\mu_n} B_{n,m}e^{-\omega_n z}z^{m-1},$$

where the $B_{n,m}$ are arbitrary constants. Similar results have previously been obtained by H. R. Pitt [Proc. Cambridge Philos. Soc. **43**, 153–163 (1947); these Rev. **9**, 40] for the equation

$$\sum_{r=0}^{R}\int_{-\infty}^{\infty} f^{(r)}(x-\lambda)dk_r(\lambda)=g(x),$$

in which $g(x)$ and $f(x)$ are functions of the real variable x. However, Yagi's theorem cannot be obtained easily from Pitt's results since the latter are based on real variable techniques not directly applicable to analytic functions.

R. H. Cameron (Minneapolis, Minn.).

Ostrom, Theodore G. The solution of linear integral equations by means of Wiener integrals. Bull. Amer. Math. Soc. **55**, 343–354 (1949).

Under suitable restrictions on the kernel $K(t, s)$ and its partial derivative with respect to t, and on the given function $z(t)$ and its derivative $z'(t)$, the author gives an explicit expression for the solution of the integral equation

$$z(t)=x(t)+\int_0^1 K(t, s)x(s)ds$$

in terms of Wiener integrals. Adding a further condition on the kernel, the author then obtains explicit formulas for the Fredholm determinant and resolvent kernel, again in terms of Wiener integrals. The proof uses results obtained by Cameron and the reviewer on transformations of Wiener integrals under translations and other linear transformations. The author mentions the possibility of using his results on the one hand to obtain approximate solutions of the integral equation which may converge faster than the Fredholm solution, and on the other hand to evaluate Wiener integrals in terms of the known (Fredholm) solution of the integral equation. *W. T. Martin.*

Carrier, G. F. A generalization of the Wiener-Hopf technique. Quart. Appl. Math. **7**, 105–109 (1949).

The partial differential equation $\phi_{yy}+y^m(\phi_{xx}+k^2\phi)=0$ is solved subject to the boundary condition that $\phi(0, y)=0$ on the line $0<b<y$. Here m is a positive integer. Further $\phi(x, y)$ satisfies certain conditions at infinity. This boundary value problem may be formulated as an integral equation of the form

$$(*)\qquad \phi(x, y)=\phi_0(x, y)+\int_a^\infty K(x, 0, y, y_0)f(y_0)dy_0,$$

where $\phi_0(x, y)$ is known and the function $K(x, x_0, y, y_0)$ is given as a definite integral which contains in the integral a product of Hankel functions. By applying the transformation

$$\bar{f}(z)=\int_0^\infty z^\nu s^{1-\nu}H_\nu^{(2)}(sz)f(s)ds,$$

where $\nu=(2+m)^{-1}$, to (*) when $x=0$ it is found that the equation (*) breaks down to an equation in z. This equation in z may be factored in a fashion similar to the one indicated by Wiener and Hopf [Paley and Wiener, Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, New York, 1934, chapter IV]. Instead of employing analyticity in half planes (as Wiener and Hopf do) there are now angular sections with vertices at the origin in which $\bar{f}(z)$ is analytic. *A. E. Heins.*

Colombo, Giuseppe. Sopra l'equazione integrale, a nucleo dipendente dal parametro, delle vibrazioni normali di una sfera immersa in un fluido. Rend. Sem. Mat. Univ. Padova **17**, 29–38 (1948).

The problem leads to the equation

$$r^2u''(r)+2ru'(r)-2u(r)=r^2\lambda^2u(r),$$

$0\leqq r\leqq 1$, with boundary conditions

$$u'(1)+[h+\alpha\lambda^2/(1+n\lambda)]u(1)=0,\ u(0)=0.$$

This system is reduced to a linear integral equation whose kernel is a rational function of λ. The kernel is symmetrizable and may be transformed into one which is positive definite. In the general case, the characteristic numbers have their real parts negative while in the limiting cases $\alpha=0$, $\alpha=\infty$, they are pure imaginary. *I. A. Barnett.*

Praporgescu, N. Sur une classe d'équations intégrales et leurs applications à la statistique. Bull. Math. Phys. Éc. Polytech. Bucarest **10** (1938–39), 64–103 (1940); **11**, 140–150 (1940).

The author considers (A) integral equations of the form $\phi(s)-\int_a^b K(t)\phi(s+t)dt=f(s)$, where f is known and either ϕ or K may be unknown, (B) the problem of minimizing an integral $\int_A^B [u(s)-\int_a^b K(t)\phi(s+t)dt]^2ds$ by a suitable choice of K, u and ϕ being given. For (B), he finds that K must satisfy a certain integral equation of the first kind. The results are stated to have applications to statistics.

G. E. H. Reuter (Manchester).

Maravall, Darío. Uniqueness theorem for nonlinear integral equations. Revista Mat. Hisp.-Amer. (4) **8**, 143–147 (1948). (Spanish)

The author considers a nonlinear integral equation of the form

$$(*)\qquad \phi(x)=\lambda\int_0^1 F(x, y)f(y, \phi(y))dy,\qquad 0\leqq x\leqq 1,$$

when, in particular, $f(y, \phi)$ satisfies a Lipschitz condition $|f(y, \phi_1)-f(y, \phi_2)|\leqq C|\phi_1-\phi_2|$, on $0\leqq y\leqq 1$, $|\phi|\leqq 1$. He purports to show that (*) has a unique solution $\phi(x)$ satisfying $|\phi(x)|\leqq 1$ on $0\leqq x\leqq 1$, whenever λ is sufficiently small in absolute value. The author does not state specifically all the properties of $F(x, y)$ and $f(y, \phi)$ used in the argument, and certain errors in reasoning occur. The paper contains no references to earlier literature on the subject.

W. T. Reid (Evanston, Ill.).

Maravall, Darío. The principle of superposition and the integration of integro-differential equations with partial derivatives. Euclides, Madrid **8**, 345–347 (1948). (Spanish)

This note deals with the use of the method of separation of variables for the determination of particular solutions of two types of integro-differential equations. In each case there is an error in the formula given for the kernel of the resulting Volterra integral equation of the second kind.

W. T. Reid (Evanston, Ill.).

Chandrasekhar, S., and Breen, Frances H. On the radiative equilibrium of a stellar atmosphere. XXIII. Astrophys. J. **107**, 216–219 (1948).

[Paper XXII appeared in the same J. **107**, 48–72, 188–215 (1948); these Rev. **9**, 593.] In continuation of paper XIX of this series [same J. **106**, 143–144 (1947); these Rev. **9**, 444], the authors tabulate the functions $H^{(0)}(\mu)$ and $H^{(1)}(\mu)$ governing the law of diffuse reflection by a semi-infinite plane-parallel atmosphere with scattering in accordance with the phase function $\lambda(1+x\cos\theta)$. For the theoretical discussion of this case, see paper XIV of the series [same J. **105**, 164–203 (1947); these Rev. **8**, 467].

F. Smithies (Cambridge, England).

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. XXIV. Astrophys. J. **108**, 92–111 (1948).

[Cf. the preceding review.] In the present paper the author obtains approximate solutions of the functional equations of paper XXII for small values of the optical thickness of the atmosphere. He uses an iterative method, starting with certain simple functions, and then derives a correction term from known integral relations. The con-

struction of standard solutions in conservative cases is also considered. An appendix [by Chandrasekhar and Breen] contains tables of some auxiliary functions, related to the exponential integrals. *F. Smithies.*

Functional Analysis, Ergodic Theory

Arrighi, Gino. Contributo ad una teoria generale degli operatori. Pont. Acad. Sci. Acta **11**, 71–78 (1947).
Remarks on functions which are not everywhere defined. *I. Kaplansky* (Chicago, Ill.).

Tong, Hing. On ideals of certain topologized rings of continuous mappings associated with topological spaces. Ann. of Math. (2) **50**, 329–340 (1949).

The paper opens with a discussion of the ring $\mathfrak{C}(X, K)$ of all complex-valued continuous functions defined on a topological space X which need not satisfy any separation axioms but which is very nearly compact. By a well-known device due to E. Čech, the author reduces his problem to a discussion of $\mathfrak{C}(X, K)$, where X is a compact Hausdorff space. His main observation, that the closed ideals $\mathfrak{J}$ of $\mathfrak{C}(X, K)$ are in one-to-one correspondence with closed sets F in X, where $\mathfrak{J} \leftrightarrow \prod_{f \in \mathfrak{J}} E[x; x \in X, f(x) = 0]$, is then an immediate consequence of a theorem of M. H. Stone [Trans. Amer. Math. Soc. **41**, 375–481 (1937), theorem 85]. It is pointed out that the closed ideals of $\mathfrak{C}(X, K)$ are all principal if X is perfectly normal. Several subsidiary results are also proved. The second part of the paper deals with the ring $\mathfrak{C}(X, \mathfrak{L}(B))$ consisting of all strongly continuous mappings of X into the ring $\mathfrak{L}(B)$ of all bounded linear operators defined on a separable Banach space B with a regular basis. It is proved that, if X is perfectly normal and compact, then every two-sided closed ideal in $\mathfrak{C}(X, \mathfrak{L}(B))$ is principal and there is an isomorphic correspondence between such ideals and the closed subsets of X. The paper closes with the statement, supported by an inconclusive argument, that this result fails if B is nonseparable. *E. Hewitt.*

Ogasawara, Tôzirô. On Fréchet lattices. I. J. Sci. Hirosima Univ. Ser. A. **12**, 235–248 (1943). (Japanese)

A vector lattice L is a Fréchet lattice if there exists a metric function (quasi-norm) $\rho(x)$ defined on L with respect to which L is a Fréchet space (an F-space in the sense of S. Banach) and such that $\rho(x) \leqq \rho(y)$ if $|x| \leqq |y|$. The author discusses conditions under which a Fréchet lattice becomes a regular vector lattice in the sense of L. Kantorovitch [Rec. Math. [Mat. Sbornik] N.S. **2**(**44**), 121–165 (1937), in particular, pp. 138–143]. Further, it is shown that a regular Fréchet lattice which is at the same time a ring lattice can be made into a Banach lattice by a suitable choice of norm if and only if it is finite-dimensional. From this follows, in particular, that the spaces (S) and (s) cannot be made into Banach spaces. *S. Kakutani* (New Haven, Conn.).

Ogasawara, Tôzirô. Remarks on a vector lattice with a metric function. J. Sci. Hirosima Univ. Ser. A. **13**, 317–325 (1944). (Japanese)

The author discusses the properties of vector lattices R_3 and R_4 introduced by L. Kantorovitch [reference in the preceding review] (the condition $\rho(x) < \rho(y)$ if $0 \leqq x < y$ is replaced here by a weaker one $\rho(x) \leqq \rho(y)$ if $0 \leqq x < y$). It is shown that an R_3-space X is an R_4-space if and only if it is possible to introduce an equivalent metric function on X for which X is a Fréchet space or a Fréchet lattice [see the preceding review]. *S. Kakutani* (New Haven, Conn.).

Ogasawara, Tôzirô. Commutativity of Archimedean semi-ordered groups. J. Sci. Hirosima Univ. Ser. A. **12**, 249–254 (1943). (Japanese)

The author gives a proof to the conjecture of G. Birkhoff [Ann. of Math. (2) **43**, 298–331 (1942), in particular, p. 329; these Rev. **4**, 3] that a conditionally complete l-group is commutative. The same conjecture was proved by K. Iwasawa [Jap. J. Math. **18**, 777–789 (1943); these Rev. **7**, 373] by using the spectral resolution. The author's proof is straightforward and algebraic. The author then proves that every Archimedean partially ordered group is commutative by showing that every such group can be embedded, preserving both group and lattice operations, into a conditionally complete l-group [A. H. Clifford, Ann. of Math. (2) **41**, 465–473 (1940); these Rev. **2**, 4]. *S. Kakutani.*

Maeda, Fumitomo, and Ogasawara, Tôzirô. Representation of vector lattices. J. Sci. Hirosima Univ. Ser. A. **12**, 17–35 (1942). (Japanese)

Ogasawara, Tôzirô. Remarks on representation of vector lattices. J. Sci. Hirosima Univ. Ser. A. **12**, 217–234 (1943). (Japanese)

Representation of vector lattices L by real-valued functions is discussed by using the method of characteristic families (spectral resolution) due to H. Freudenthal [Nederl. Akad. Wetensch., Proc. **36**, 641–651 (1936)] and F. Wecken [Math. Z. **45**, 377–404 (1939); these Rev. **1**, 12].

For any subset A of L, let $A^\perp$ be the set of x such that $|x| \wedge |y| = 0$ for all $y \in A$. Then $P = P(x)$ is a principal ideal generated by x if $P = \{x\}^{\perp\perp}$; $e \in L$ is a unit if $P(e) = L$, i.e., if $\{e\}^\perp = \{0\}$; and $N \subset L$ is a normal ideal if $N = N^{\perp\perp}$. Every principal ideal is clearly a normal ideal. If L has a unit element e, then the family $\mathfrak{P}$ of all principal ideals P of L is a distributive lattice with 0 and 1. For any $x \in L$, the characteristic family of x is given by

$$\{P_\lambda \equiv P((\lambda e - x) \vee 0) \mid -\infty < \lambda < \infty\}$$

and determines, in a natural way, a real-valued ($\pm\infty$ included) continuous function $x(\omega)$ defined on the representation space $\Omega_{\mathfrak{P}}$ of $\mathfrak{P}$ in the sense of H. Wallman [Ann. of Math. (2) **39**, 112–126 (1938)]. Then $x \to x(\omega)$ is a linear and lattice homomorphism of L into the vector lattice of real-valued ($\pm\infty$ included) continuous functions on $\Omega_{\mathfrak{P}}$ which becomes an isomorphism if L is Archimedean, i.e., if $\lambda x \to 0$ ((o)-convergence) as $\lambda \to 0$ for every fixed element $x \in L$. (Note that every Archimedean vector lattice L is embeddable into a complete vector lattice [A. H. Clifford, Ann. of Math. (2) **41**, 465–473 (1940); these Rev. **2**, 4] and that in case L itself is complete or σ-complete, the representation obtained above is essentially the same as that of H. Nakano [Proc. Phys.-Math. Soc. Japan (3) **23**, 485–511 (1941); these Rev. **3**, 210].)

In general, when L does not necessarily have a unit element, the family $\mathfrak{N}$ of all normal ideals N of L is considered. Then $\mathfrak{N}$ is a complete Boolean algebra, and the realization space $\Omega_{\mathfrak{N}}$ of $\mathfrak{N}$ in the sense of M. H. Stone [Trans. Amer. Math. Soc. **41**, 375–481 (1937)] can be used instead of $\Omega_{\mathfrak{P}}$ above. If $\{P(e_\alpha)\}$ is a maximal family of principal ideals $P(e_\alpha)$ of L with the orthogonality property: $e_\alpha \wedge e_\beta = 0$ for $\alpha \neq \beta$, then, for any $x \in L$, the characteristic family of x is given by $\{P_\lambda \equiv \mathsf{V}_\alpha(P((\lambda e_\alpha - x) \vee 0) \wedge P(e_\alpha)) \mid -\infty < \lambda < \infty\}$.

If L is a complete (σ-complete) vector lattice, then L is Archimedean and hence the representation obtained above is certainly an isomorphism; but this representation is not necessarily a complete isomorphism (σ-isomorphism). If, however, we consider a certain L-valued completely (countably) additive measure μ defined on the Borel field of all Borel subsets (Baire subsets) of $\Omega_{\mathfrak{R}}$, and if we neglect those subsets of $\Omega_{\mathfrak{R}}$ whose μ-measure is zero (i.e., if we identify two functions which differ from each other on a set of μ-measure zero), then the representation thus reduced will become a complete isomorphism (σ-isomorphism). Thus the discussion of the Stone representation spaces of complete or σ-complete Boolean algebras becomes important [see the following review].

Finally, the problem of finding a linear and lattice isomorphic representation of L by real-valued ($\pm\infty$ excluded) functions is discussed. Previously, T. Nakayama [Proc. Imp. Acad. Tokyo **18**, 185–187 (1942); these Rev. **7**, 236] gave an example of a vector lattice for which this problem is impossible. It is now shown that this is the case for most of the usual Banach lattices. It is proved that in order for this problem to be possible for a regular vector lattice L of type K_δ^- [L. Kantorovitch, Rec. Math. [Mat. Sbornik] N.S. **7**(**49**), 209–284 (1940); these Rev. **2**, 317] it is necessary and sufficient that L be a generalized sequence space, i.e., that there exists a (not necessarily countable) family $\{e_\alpha\}$ of elements e_α of L with the orthogonality property $(e_\alpha \wedge e_\beta = 0$, for $\alpha \neq \beta)$ such that every $x \in L$ is expressible in the form $x = \mathsf{V}_\alpha \lambda_\alpha e_\alpha$, where λ_α is real and equal to 0 except possibly for a countable number of α. Thus the problem is impossible for (L^p), $1 \leqq p < \infty$. *S. Kakutani* (New Haven, Conn.).

Ogasawara, Tôzirô. Theory of vector lattices. I. J. Sci. Hirosima Univ. Ser. A. **12**, 37–100 (1942). (Japanese)
Ogasawara, Tôzirô. Theory of vector lattices. II. J. Sci. Hirosima Univ. Ser. A. **13**, 41–161 (1944). (Japanese)

Systematic and detailed investigations on the structure and properties of vector lattices are carried out. The basic idea is to use the concrete representation of vector lattices by real-valued functions as developed by the author (partly jointly with F. Maeda) in two previous papers [see the preceding review]. The paper consists of five parts: (1) Boolean algebras and their representation spaces, (2) semi-ordered linear spaces, (3) Kantorovitch spaces, (4) integration of vector-lattice-valued functions, (5) integral representation of linear operations.

Part 1 starts with the definition and discussion of fundamental properties of vector lattices and Boolean algebras. Characterizations of representation spaces of complete or σ-complete Boolean algebras in the sense of M. H. Stone [Trans. Amer. Math. Soc. **41**, 375–481 (1937)] are obtained. A compact Hausdorff space Ω is a Stone representation space of a complete (σ-complete) Boolean algebra $\mathfrak{B}$ if and only if one of the following five mutually equivalent conditions is satisfied: (1) the closure of every open set (open F_σ-set) is open, (2) every Borel set (Baire set) is equivalent to an open-closed set (two subsets of Ω are equivalent if their symmetric difference is a set of first category), (3) every Borel measurable (Baire measurable) real-valued function on Ω is equivalent to a continuous function (two functions defined on Ω are equivalent if they differ from each other only on a set of first category), (4) the family of all open-closed subsets of Ω is a basis of Ω (i.e., Ω is a totally disconnected space) and forms a complete (σ-complete) Boolean algebra isomorphic to $\mathfrak{B}$, (5) the family of all real-valued continuous functions on Ω (which may take the values $\pm\infty$, but only on a nowhere dense subset of Ω) is a complete (σ-complete) vector lattice L. In case $\mathfrak{B}$ is a complete Boolean algebra, the family $\mathfrak{N}$ of all normal ideals N of L is isomorphic to $\mathfrak{B}$, and (5) may be considered as a representation of a complete vector lattice L by real-valued continuous functions defined on the Stone representation space $\Omega_{\mathfrak{R}}$ of $\mathfrak{N}$ as discussed in a previous paper of the author [see the preceding review]. These results are then applied to show that every Archimedean vector lattice can be embedded into a complete vector lattice [A. H. Clifford, Ann. of Math. (2) **41**, 465–473 (1940); these Rev. **2**, 4] and that every Archimedean ring lattice is commutative [see the second preceding review]. The results obtained in part 1 overlap largely with those published recently by M. H. Stone [see the following review].

Parts 2 and 3 are devoted to the study of Banach lattices. The condition (i) $\|x\| \leqq \|y\|$ if $|x| \leqq |y|$ is always assumed. Modifying the notion of B_2-space of L. Kantorovitch [Rec. Math. [Mat. Sbornik] N.S. **2**(**44**), 121–165 (1937)] and K_2-space of L. Kantorovitch and B. Vulich [Compositio Math. **5**, 119–165 (1937)], a Banach lattice is called a K-space if the following conditions are satisfied: (ii) $\|x_n\| \to 0$ if $x_n \downarrow 0$, (iii) $\sup_n x_n$ exists if $x_n \leqq x_{n+1}$, $n = 1, 2, \cdots$ and if $\lim_n \|x_n\| < \infty$. [Note that (ii)+(iii) is equivalent to (iv) $\lim_n x_n$ (strong limit) exists if $x_n \leqq x_{n+1}$, $n = 1, 2, \cdots$ and if $\lim_n \|x_n\| < \infty$.] All (L^p)-spaces and (l^p)-spaces $(1 \leqq p < \infty)$ are K-spaces. The K-spaces are characterized as weakly complete Banach lattices. In a K-space every finite order interval $\{x \mid a \leqq x \leqq b\}$ is sequentially weakly compact; a Banach lattice L is reflexive if and only if both L and its conjugate space $\bar{L}$ are K-spaces; a Banach lattice is reflexive if and only if it is locally sequentially weakly compact. [Recently, W. Eberlein, Proc. Nat. Acad. Sci. U. S. A. **33**, 51–53 (1947); these Rev. **9**, 42, proved this result for general Banach spaces.] It is also proved that a Banach lattice is reflexive whenever its second conjugate space is separable. [Note that the same thing for general Banach spaces is still an open conjecture.]

Further, a Banach lattice is called a Bochner lattice [Proc. Nat. Acad. Sci. U. S. A. **26**, 29–31 (1940); these Rev. **1**, 239] if there exists a sequence of positive (o)-continuous linear functionals $f_k(x)$ defined on it such that (v) $\lim_n x_n$ (strong limit) exists if $x_n \leqq x_{n+1}$, $n = 1, 2, \cdots$ and if $\lim_n f_k(x_n) < \infty$ for $k = 1, 2, \cdots$. The space (L) is a Bochner lattice. The properties of Bochner lattices are discussed. It is shown that a reflexive Bochner lattice is a sequence space, and a condition for a sequence space to be a reflexive Bochner lattice is obtained. Abstract ergodic theorems are formulated and proved in K-spaces and Bochner lattices. For example, if $T(x)$ is a bounded linear transformation of a K-space into itself such that $\{T^n(x) \mid n = 1, 2, \cdots\}$ is (o)-bounded for each x, then the sequence $n^{-1}\sum_{k=1}^{n} T^k(x)$ is (o)-convergent (and hence strongly convergent) for each x. Finally, abstract (L^p)-spaces $(1 \leqq p < \infty)$, abstract (S)- and (s)-spaces are defined and their properties discussed.

Part 4 deals with the theory of integration of vector-lattice-valued functions. For any vector lattice L, an L-valued function $x(\omega)$ (defined on a certain measure space Ω) is (o)-measurable if it is an (o)-limit almost everywhere of a sequence $\{x_n(\omega) \mid n = 1, 2, \cdots\}$ of L-valued measurable simple (finite-valued) functions, and the (o)-integral of $x(\omega)$ is defined as the (o)-limit (if it exists) of the integral of each $x_n(\omega)$ whose definition is clear. The (o)-integral in K-spaces is important in the discussion of the integral rep-

resentation of linear operations as discussed in part 5. In case L is a K-space, the (o)-integral of the author is stronger than the Dunford integral [Trans. Amer. Math. Soc. **37**, 441–453 (1935)], but weaker than Bochner integral [Fund. Math. **20**, 262–276 (1933)]. Finally, vector-lattice-valued set functions are considered, their properties (countable additivity and absolute continuity) are discussed, and theorems of Radon-Nikodym type are obtained: in case L is an abstract (S)-space, a Bochner lattice, or an abstract (L)-space, the indefinite (o)-integral is characterized as an L-valued (o)-absolutely-continuous countably additive set function.

In part 5, integral representation of linear operations in vector lattices is discussed. Generalizing a result of F. Riesz [C. R. Acad. Sci. Paris **149**, 974–977 (1909)] concerning the integral representation of positive linear functionals on the space $C(\Omega)$ of real-valued continuous functions $x(\omega)$ defined on a compact Hausdorff space Ω, it is shown that, for any regular vector lattice L in the sense of L. Kantorovitch [Rec. Math. [Mat. Sbornik] N.S. **7(49)**, 209–284 (1940); these Rev. **2**, 317] every L-valued (o)-bounded positive linear operation $T(x)$ defined on $C(\Omega)$ can be represented as an integral of $x(\omega)$ with respect to a certain L-valued countably additive (nonnegative-valued) measure defined on the Borel field of all Baire subsets of Ω. This result is then applied to show that an (o, b)-continuous linear operation (i.e., a linear transformation $T(x)$ of a Banach space X to a regular Banach lattice L which maps norm-bounded sets of X into order-bounded sets of L) maps weakly convergent sequences of X into (o)-convergent sequences of L. Thus, if X is locally weakly compact [i.e., by a result of W. Eberlein, if X is reflexive] then T is completely continuous. Further, the integral representation of operators in (L^p), $1 \leqq p < \infty$, is discussed in detail. The results of N. Dunford and B. J. Pettis [Trans. Amer. Math. Soc. **47**, 323–392 (1940); these Rev. **1**, 338] concerning the integral representation of completely continuous or weakly completely continuous linear operations in (L) are generalized in various ways. *S. Kakutani.*

Stone, M. H. Boundedness properties in function-lattices. Canadian J. Math. **1**, 176–186 (1949).

The present results [announced in Proc. Nat. Acad. Sci. U. S. A. **26**, 280–283 (1940); these Rev. **1**, 338] concern the topological properties of a topological space X which accompany the following property $A_{\aleph}$ of the lattice C of continuous real-valued functions on X, where $\aleph$ is any infinite cardinal number: every nonvoid subset of C with an upper bound in C and having at most $\aleph$ members has a least such upper bound. If $A_{\aleph}$ is assumed for a completely regular X then (1) $E_f(\lambda)^-$ is open for every f in C ($E_f(\lambda)$ is the "spectral" set of f on which $f(x) < \lambda$); (2) the closed-open sets form a basis for X; (3) the union of at most $\aleph$ closed-open sets has an open closure; (4) for f in C, $E_f(\lambda)$ is the union of at most $\aleph$ of open-closed sets. (Indeed $E_f(\lambda) = \cup E_f(\mu)^-$, where $\mu \subset \lambda$ and $\mu \varepsilon M$ and M is any dense subset of the line. This set may be chosen as countable. The technique and application of the spectral sets $E_f(\lambda)$ is presented.) Conversely (3) and (4) for a completely regular X imply $A_{\aleph}$. When the cardinal number of C is at most $\aleph$ or when $\aleph = \aleph_0$ and X is normal, simpler statements are possible. For compact Hausdorff spaces X, $A_{\aleph}$ holds if and only if X is the representative Boolean space for an $\aleph$-additive Boolean algebra. Turning to Baire functions f defined on X, it is shown that any such f which is bounded above and below by members of C differs from a suitable g in C on at most a set of first category. This g is necessarily unique if X is such that all first category sets have void interiors.

R. Arens (Los Angeles, Calif.).

Maeda, Fumitomo. The center of lattices. J. Sci. Hirosima Univ. Ser. A. **12**, 11–15 (1942). (Japanese)

The equivalence of the following four conditions for an element a of a (reducible or irreducible) continuous geometry L was proved by J. von Neumann [Continuous Geometry, parts I and III, Institute for Advanced Study, Princeton, 1936, 1937]: (1) a belongs to the center of L, (2) a is a neutral element of L, (3) there exists a complement a' of a such that $x = (x \wedge a) \vee (x \wedge a')$ for any $x \varepsilon L$, (4) the complement of a is unique. In a previous paper [same J. Ser. A. **11**, 1–6 (1941); these Rev. **2**, 318] the author proved the same result for a lattice of projections in a Hilbert space. In the present paper this result is proved for a general class of lattices which contains both continuous geometry and lattice of projections as a special case. *S. Kakutani.*

Maeda, Fumitomo. Dimension lattice of reducible geometries. J. Sci. Hirosima Univ. Ser. A. **13**, 11–40 (1944). (Japanese)

It was shown by I. Halperin [Ann. of Math. (2) **40**, 581–599 (1939); these Rev. **1**, 45] and J. von Neumann [Continuous Geometry, parts I and III, Institute for Advanced Study, Princeton, 1936, 1937] that the set of all equivalence classes of mutually "equi-dimensional" elements of a (reducible or irreducible) continuous geometry L is a complemented modular lattice $D(L)$ on which "addition" is defined for certain pairs of elements. The properties of $D(L)$, called the dimension lattice of L, are discussed in detail. The lattice $D(L)$ is "one-dimensional" if and only if L is irreducible. The embedding of $D(L)$ in a vector lattice, the decomposition of $D(L)$ into direct sums, and its relation with the decomposition of L into irreducible geometries are discussed. Similar problems were also discussed by T. Iwamura [Jap. J. Math. **19**, 57–71 (1944); these Rev. **8**, 35] by a different method. *S. Kakutani.*

Motchane, Léon. Sur la représentation paramétrique des ensembles de vecteurs dans l'espace de Hilbert. C. R. Acad. Sci. Paris **228**, 353–355 (1949).

This paper states, without proof, certain results on the set of points of continuity of a Baire function of the first class, defined on a k-dimensional space and with values in a Hilbert space. For such functions weak and strong convergence properties of a sequence at a point are inferred from the properties of the limit function. When the members of the sequence are continuous, the set of points of continuity of the limit can be resolved into sets associated with these convergence properties. A further discussion is indicated on the set of "lines of continuity" parallel to the coordinate axes. *F. J. Murray* (New York, N. Y.).

Dixmier, Jacques. Position relative de deux variétés linéaires fermées dans un espace de Hilbert. Revue Sci. **86**, 387–399 (1948).

The author makes a detailed study of the system consisting of a pair M, N of closed subspaces in Hilbert space. It is easy to reduce the study of the general case to that in which M and N are in "position p"; that is,

$$M \cap N = M^{\perp} \cap N = M \cap N^{\perp} = M^{\perp} \cap N^{\perp} = 0.$$

A principal result asserts that it is possible to associate with each pair M, N in position p a self-adjoint operator A in such a manner that two systems are unitary equivalent if and only if this is true of the associated self-adjoint operators. A self-adjoint operator is associated with some system if and only if its bounds are zero and one, and neither of these numbers is in its point spectrum. Thus the unitary equivalence problem for systems M, N is reduced to that for self-adjoint operators and so completely solved. Other results involve the ring of operators generated by two projections and a notion of bisection for pairs of closed subspaces. If P_M and P_N are projections whose ranges M and N are in position p then every operator in the ring generated by P_M and P_N is of the form $F(P_M+P_N)+S_VG(P_M+P_N)$, where F and G are Borel functions and S_V is the unique self-adjoint involution which interchanges M and N and is such that $S_V(x, x)>0$ for $x\in M\cup N$. The null spaces V of S_{V-1} and $V^\perp$ of S_{V-1} are called the interior and exterior bisectors of the pair M, N and have various properties which the author investigates. As an application of the theory developed in this paper relatively elementary proofs are given for the decomposition of an operator into unitary and self-adjoint factors and for the decomposition of a self-adjoint operator into its positive and negative parts.

G. W. Mackey (Chicago, Ill.).

Morita, Kiiti. On normal anti-linear transformations in Hilbert space. Jap. J. Math. **19**, 363–373 (1947).

Let J be a conjugation in Hilbert space $\mathfrak{H}$, i.e., $(Jf, Jg)=(g, f)$ and J has range and domain $\mathfrak{H}$. To a certain extent, J is to be regarded as fixed in this paper. To every linear transformation $[A]=AJ$. A conjugate linear transformation A is normal if $JAA^*J=A^*A$. The canonical resolution $A=H[U]=[U]H$ for normal operators is obtained, where H is definite and U is unitary. Then U in turn is expressed in terms of $P=\frac{1}{2}(U+JU^*J)$ and $Q=\frac{1}{2}(U-JU^*J)$; $[P]$ and $[Q]$ in turn are normal and further resolved. For instance, $[Q]=H_Q[W_Q]$. The reality character of $[A]$ on the range and zeros of $[W_Q]$ is given in detail. The motivation is stated to be associated with the fact that in finite-dimensions the complex roots of a "real" transformation occur in conjugate pairs but more elegant elementary methods are preferred by the author. The final result of the paper is that $[A_1]$ and $[A_2]$ are unitarily equivalent if and only if the linear transformations $[A_1]^2$ and $[A_2]^2$ are unitarily equivalent.

F. J. Murray (New York, N. Y.).

Julia, Gaston. Sur la permutabilité et l'antipermutabilité des opérateurs hermitiens. C. R. Acad. Sci. Paris **228**, 962–964 (1949).

The author discusses some elementary implications among relations of the forms $AB^n=B^nA$ and $AB^m=-B^mA$ for bounded Hermitian operators A and B.

P. R. Halmos.

Svirskiĭ, I. V. The determination of inverse operators by certain properties of the direct operators. Doklady Akad. Nauk SSSR (N.S.) **63**, 103–106 (1948). (Russian)

A simple method is given for solving the following extension problem in Hilbert space $\mathfrak{H}$. Let A be a bounded generalized Hermitian operator whose domain $M=\mathfrak{D}A$ is a closed subspace of $\mathfrak{H}$ [see F. J. Murray, Trans. Amer. Math. Soc. **37**, 301–338 (1935)]. The operator A is supposed to be nonnegative, i.e., $(Af, f)\geqq 0$ for all $f\in M$. The problem is to find all nonnegative bounded self-adjoint extensions $\tilde{A}$ of A to the whole space $\mathfrak{H}$. To do this, denote by N the orthogonal complement of M in $\mathfrak{H}$, and consider a self-adjoint bounded extension $\tilde{A}_\epsilon$ in $\mathfrak{H}$ of $A_\epsilon=A+\epsilon E$, where $\epsilon>0$ and E is the identity transformation in M. Split $\tilde{A}_\epsilon$ into four operators $A_{0,\epsilon}$, B, B^*, X_ϵ with domains M, M, N, N and ranges included in M, N, M, N, respectively, where $A_{0,\epsilon}$, X_ϵ are bounded self-adjoint, $A_{0,\epsilon}$ is nonnegative and B^* is the Murray adjoint [loc. cit.] of $B=\mathrm{Proj}_N A$. If $h\in\mathfrak{H}$ and $h=f+g$ with $f\in M$, $g\in N$, we have

$$(\tilde{A}_\epsilon h, h)=(A_{0,\epsilon}[f+A_{0,\epsilon}^{-1}B^*g], [f+A_{0,\epsilon}^{-1}B^*g]) + ([-BA_{0,\epsilon}^{-1}B^*+X_\epsilon]g, g).$$

It follows that the nonnegativeness of A_ϵ is equivalent to that of $(-BA_{0,\epsilon}^{-1}B^*+X_\epsilon)$. Letting $\epsilon\to 0$, we obtain the most general required extension $\tilde{A}$ of A as $[A_0, B, B^*, X]$, where $X=\lim_{\epsilon\to 0} BA_{0,\epsilon}^{-1}B^*+C$, $A_0=\mathrm{Proj}_M A$, and C is any nonnegative bounded self-adjoint operation in N. This can be generalized to the case of unbounded self-adjoint operations. [For non-self-adjoint operations see M. G. Kreĭn and M. A. Krasnosel'skiĭ, Uspehi Matem. Nauk (N.S.) **2**, no. 3(19), 60–106 (1947); these Rev. **10**, 198.] The author applies his result to the particular case where M has finite dimension in connection with a problem which is not clearly formulated.

O. M. Nikodým (Gambier, Ohio).

Fage, M. K. The symmetry theorem for Hermitian operators. Mat. Sbornik N.S. **24(66)**, 107–117 (1949). (Russian)

Using the results and terminology of Plesner [Uspehi Matem. Nauk (N.S.) **1**, no. 1(11), 192–216 (1946); these Rev. **9**, 43] and Plesner and Rohlin [ibid., 71–191 (1946); these Rev. **9**, 43] the author considers the family of subspaces $\mathfrak{H}$ left invariant by a Hermitian [self-adjoint?] operator. In terms of the so-called spectral or Hellinger type [above reference] a spectral norm $|\mathfrak{H}|$ is defined for the invariant subspaces $\mathfrak{H}$ and some formal properties of this norm are established, e.g., the following symmetry theorem. If $\mathfrak{H}_i$ are invariant subspaces and H_i their orthogonal projections $(i=1, 2)$ then $|H_1\mathfrak{H}_2|=|H_2\mathfrak{H}_1|$. Next a scalar product between invariant subspaces is defined in terms of the spectral type and is shown to have some of the formal properties of the ordinary scalar product. Using these the paper closes with the establishment of a formal analogue of the orthogonalization process.

F. I. Mautner.

Friedrichs, K. O. On the perturbation of continuous spectra. Communications on Appl. Math. **1**, 361–406 (1948).

The perturbation theory for linear operations with a discrete spectrum was treated extensively by F. Rellich [Math. Ann. **113**, 600–619 (1936); 677–685 (1937); **117**, 356–382 (1940); **118**, 462–484 (1942); these Rev. **2**, 105; **6**, 71]. The present paper deals with the case of a continuous spectrum. The starting point of the investigation is the theory of "spectral representation" (rather than spectral resolution) developed by the author in an earlier paper [Math. Ann. **115**, 249–272 (1938)]. The problem treated first deals with the perturbation of an operator L having a simple continuous spectrum consisting of an interval I of the λ-axis by an operator of the form ϵK, where K is a Hermitian integral operator (independent of ϵ) whose "kernel" $k(\lambda, \mu)$ satisfies a certain Lipschitz condition and is zero if λ or μ is at the boundary of the interval I. It is shown that for small enough ϵ the operator $L+\epsilon K$ possesses also a continuous spectrum covering the interval I. It is also

shown that if the above boundary condition for $k(\lambda, \mu)$ is not satisfied, K can be chosen so that $L+\epsilon K$ has a point eigenvalue for arbitrarily small ϵ. Next the case is treated where the undisturbed operator L has a point eigenvalue and a disturbing operator F is constructed in such a way that the operator $L+\epsilon F$ has, for small enough ϵ, no point eigenvalue at all. In this case the transformations leading to the "spectral representation" of $L+\epsilon F$ do not depend analytically on ϵ; consequently, the usual formal perturbation procedure (expansion according to powers of ϵ) breaks down.

The operators $L+\epsilon K$ and $L+\epsilon F$ are then identified with the energy operator H in the Schroedinger equation $\partial\psi/\partial t=-iH\psi$ for the "state function" $\psi=\psi(t)$ and the asymptotic behavior of ψ for $t\to\pm\infty$ is studied. The significance of the results obtained for the quantum theory and especially their relation to some work of Weisskopf and Wigner [Z. Physik **63**, 54–73 (1930); **65**, 18–29 (1930)] and of Heisenberg [ibid. **120**, 513–538, 673–702 (1943); these Rev. **4**, 292] is discussed. The paper closes with statements about a different approach (using Fourier transforms) to a class of perturbation problems. The details of this new method are left for a future publication. *E. H. Rothe.*

von Neumann, John. On rings of operators. Reduction theory. Ann. of Math. (2) **50**, 401–485 (1949).

If one attempts to study noncommutative families, such as groups or rings, of operators in Hilbert space one is faced with the necessity of generalizing the notion of direct sum of vector spaces. This is done in this paper [which was written in 1938]. Given a measure space of points λ, which is here the real line with finite Lebesgue-Stieltjes measure $d\sigma(\lambda)$, and for each λ a separable Hilbert (or finite-dimensional) space $\mathfrak{H}_\lambda$, then the generalized direct sum of the spaces $\mathfrak{H}_\lambda$ is defined to be a certain family of vector-valued functions $f(\lambda)$ with values in the various $\mathfrak{H}_\lambda$. Axioms are imposed so that the inner product $(f(\lambda), g(\lambda))$ is always σ-integrable. Part I deals first with this definition and, after some preliminaries, with the construction of a complete orthonormal system in each $\mathfrak{H}_\lambda$ in a measurable fashion. This leads to an equivalent definition of generalized direct sum, now described by means of sequences $x_m(\lambda)$ of numerical measurable functions satisfying $\int\sum|x_m(\lambda)|^2d\sigma(\lambda)<\infty$. In terms of a given generalized direct sum a certain spectral family $E(\mu)$ is now defined and it is shown how $\sigma(\lambda)$ and the $E(\mu)$ determine the generalized direct sum. This leads to part II where a certain Abelian ring P of operators in $\mathfrak{H}$ is introduced and after a natural definition of equivalence of generalized direct sums it is shown that this ring P determines the generalized direct sum uniquely up to equivalence. Now the decomposition of single operators A in $\mathfrak{H}$ can be studied.

Part III deals with the decomposition of rings M of operators, always assumed to be self-adjoint and weakly closed. This requires apparently more powerful tools and much more difficult arguments than the preceding parts. Above all the author has to establish the possibility of applying the axiom of choice in a measurable fashion. This is then used to study the notion of measurability of functions $M(\lambda)$ whose value is for each λ a ring of operators in the space $\mathfrak{H}$. Part III culminates in a theorem which states the decomposability properties of a ring M and their relation with the fixed ring P.

In part IV the case where P is the center of M is considered. In this case the rings $M(\lambda)$ obtained from decomposing M are factors (i.e., their centers are the scalars) and it is shown that such a decomposition of M into factors is always possible and essentially unique. This is a generalization of the well-known decomposition theorems on semi-simple rings with chain-condition. In the last sections the question is studied whether the λ-sets for which $M(\lambda)$ is of given type are measurable. The answer is shown to be affirmative except for the distinction between the so-called types II_∞ and III. For this last case the measurability question is left open and, at least in the author's mode of attack, leads to a problem shown by Gödel to be unsolvable in the conventional systems of logic. These last sections also contain an expression of certain weight functions on M as integrals over the relative dimension functions on the factors $M(\lambda)$. Thus the study of weakly closed self-adjoint algebras of operators in separable Hilbert space is reduced to a large extent to the study of factors. *F. I. Mautner.*

Godement, Roger. Théorie générale des sommes continues d'espaces de Banach. C. R. Acad. Sci. Paris **228**, 1321–1323 (1949).

Let Ω be a locally compact space and let there be given for each $\zeta\in\Omega$ a Banach space $H(\zeta)$. Let F denote the set of all functions x with domain Ω and range in the union of the $H(\zeta)$ such that $x(\zeta)\in H(\zeta)$ for all $\zeta\in\Omega$. The continuous sums with which the author is concerned are vector spaces whose elements are members of F or equivalence classes of members of F. In the first section the functions dealt with are such that $\|x(\zeta)\|$ is a continuous function of ζ. In the last section they are such that $\|x(\zeta)\|^p$ is summable with respect to a measure in Ω for some $p\geqq 1$. In both cases they are also such as to be approximable in an appropriate sense by the members of certain other classes of functions. The results announced have to do with the expected elementary properties of the spaces obtained. As the author remarks, the case in which Ω is the real line, p is 2 and all of the $H(\zeta)$ are Hilbert spaces has been treated by J. von Neumann [cf. the preceding review]. In the middle section the author presents an example of a continuous sum derived from a normed ring with involution A. Here Ω is the compact space of all positive linear functionals on A and the $H(\zeta)$ are the Hilbert spaces associated with the corresponding unitary representations of the ring A. *G. W. Mackey* (Chicago, Ill.).

Karlin, S. Bases in Banach spaces. Duke Math. J. **15**, 971–985 (1948).

The author investigates structural properties of a Banach space which result from the space's possessing a basis. An absolute basis is defined as a sequence of elements $\{x_n\}$ in E such that for every x in E there exists a unique set of real numbers $\{a_n\}$ such that $x=\sum a_nx_n$ and this series converges unconditionally. It is shown that if E has an absolute basis and if E^* is separable, then E^* is weakly complete. On the other hand, if E has an absolute basis and if E^* is weakly complete, then E^* has an absolute basis. As a consequence the space of continuous functions has no absolute basis. The paper also contains some results on absolutely convergent bases and on the relation between bases and projections. *R. S. Phillips* (Los Angeles, Calif.).

Rothe, E. H. Gradient mappings and extrema in Banach spaces. Duke Math. J. **15**, 421–431 (1948).

The author discusses the extension of the notion of gradient mapping [Ann. of Math. (2) **47**, 580–592 (1946); these Rev. **8**, 158] to a general real Banach space. In Hilbert

space a gradient mapping is defined in terms of the Fréchet differential of a scalar function, and the author follows a similar course here. If $I(x)$ is a scalar function defined in a real Banach space E and possessing a Fréchet differential $D(x, h)$, then $D(x, h)=l_x(h)$, where l_x is a linear functional in E, i.e., $l_x \in E^*$. The mapping $x \to l_x$ is then called a gradient mapping. If this mapping is completely continuous, $D(x, h)$ is said to be completely continuous. The author's main result is that if the scalar function $I(x)$ possesses a completely continuous Fréchet differential in a convex set V, then $I(x)$ is continuous in V in the "bounded weak" topology. If, in addition, V is closed and compact in this topology, then $I(x)$ attains a maximum and a minimum in V and, if n is any positive integer, $\|x\|^n+I(x)$ attains a minimum in V. *F. Smithies* (Cambridge, England).

Monna, A. F. Sur les espaces linéaires normés. VI. Nederl. Akad. Wetensch., Proc. **52**, 151–160 = Indagationes Math. **11**, 40–49 (1949).

[Part V appeared in the same Proc. **51**, 197–210 = Indagationes Math. **10**, 68–81 (1948); these Rev. **9**, 517.] Let E be a complete totally non-Archimedean normed space over a valued field K satisfying the following conditions. (A) The set of norms $\|\xi\|$ forms a sequence $\{C_n\}$, $-\infty < n < +\infty$; (B) for every n, the set $\|\eta\| \leqq C_n$ modulo the set $\|\eta\| < C_n$ is a vector space over the residue field $\bar{K}$ of K in a natural way; let it be assumed that the dimension is 1. If $\xi_n \in E$ is such that $\|\xi_n\| = C_n$, then it is proved that every element of E is uniquely of the form $\sum_{i=0}^{\infty} a_{n-i}\xi_{n-i}$, where the a_{n-i} are taken from a system of representatives of $\bar{K}$ in K. Conversely if (A) is satisfied and this expansion holds, then (B) is also true. Again assuming (A) and (B), the author proves that E is locally compact if and only if $\bar{K}$ is finite. There is an application to Diophantine approximation in p-adic fields. *I. S. Cohen* (Cambridge, Mass.).

Kaplansky, Irving. Regular Banach algebras. J. Indian Math. Soc. (N.S.) **12**, 57–62 (1948).

The Banach algebras here are real and are not assumed to be commutative nor to possess an identity element. Regularity means that, for every element x, there exists a y such that $xyx=x$ [von Neumann, Proc. Nat. Acad. Sci. U. S. A. **22**, 707–713 (1936)]. The author proves that a regular Banach algebra is necessarily finite dimensional. This was proved for the commutative case by Arens and the author [Trans. Amer. Math. Soc. **63**, 457–481 (1948); these Rev. **10**, 7] and for rings of operators on Hilbert space by von Neumann [Continuous Geometry, part II, Institute for Advanced Study, Princeton, 1937]. It also includes the fact that the group algebra of a locally compact group is regular if, and only if, the group is finite [I. Segal, Yale dissertation, 1940]. A ring is called "weakly regular" if, for every $x \neq 0$, there exists $y \neq 0$ such that $yxy=y$. Weak regularity is equivalent to the condition that every right ideal contain a nonzero idempotent. Regularity implies weak regularity; however, there exist infinite-dimensional, weakly regular Banach algebras. The Banach algebra $C(X)$ of all real continuous functions on a compact Hausdorff space X is weakly regular if, and only if, each open set in X contains an open and closed set. *C. E. Rickart.*

Yood, Bertram. Banach algebras of bounded functions. Duke Math. J. **16**, 151–163 (1949).

Soit S un ensemble quelconque, et pour chaque $t \in S$, soit $\mathfrak{M}_t$ un espace compact. L'auteur considère l'espace de Banach $B(S)$ des fonctions complexes bornées et continues dans l'espace somme topologique des $\mathfrak{M}_t$, et détermine les idéaux maximaux de cette algèbre, ou, ce qui revient au même, les caractères de l'algèbre. Il montre que tout caractère est de la forme $f \to \lim_{\mathfrak{U}} f(t, \gamma_t)$ où $\mathfrak{U}$ est un ultrafiltre sur S et $\gamma=(\gamma_t)$ un élément de l'espace compact Γ, produit des espaces $\mathfrak{M}_t$. *J. Dieudonné* (Nancy).

Menger, Karl. The topology of the triangle inequality. Revista Ci., Lima **50**, 155–165 (1948).

The author calls real vector spaces with generalized norms satisfying only the two conditions (1) $|\alpha a| = \alpha |a|$ for $\alpha \geqq 0$ and (2) $|a+b| \leqq |a|+|b|$, triangular vector spaces. They occur in connection with regular problems in the calculus of variations which are not necessarily positive definite. The purpose of the paper is to develop the topological consequences of axioms (1) and (2) in the two-dimensional case [higher-dimensional spaces of this type are treated in a later paper, Canadian J. Math. **1**, 94–104 (1949); these Rev. **10**, 306]. The set of vectors x for which $-\rho < |x-b| < \rho$ is called the open absolute sphere of radius ρ about b. A topology for the space is obtained by taking the absolute spheres as neighborhoods. Clearly the intersection of all absolute spheres about the origin θ is the null sphere, that is, the set of all vectors with zero norm. It is proved that the null sphere of a two-dimensional triangular vector space belongs to one of seven types. This leads at once to a classification of the various topologies of the plane into elliptic, parabolic, hyperbolic, linear and three types of degenerate topologies. The conic section designations are suggested by possible shapes for the absolute spheres in these cases, and the term "degenerate" is used when the null sphere contains an open sector of the plane. *D. H. Hyers.*

Wolfowitz, J. Remarks on the notion of recurrence. Bull. Amer. Math. Soc. **55**, 394–395 (1949).

Simple proofs of Poincaré's recurrence theorem and of a theorem of Kac [Bull. Amer. Math. Soc. **53**, 1002–1010 (1947), theorem $2'$; these Rev. **9**, 194] on the mean recurrence time. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Hopf, E. Ergodic theory. Uspehi Matem. Nauk (N.S.) **4**, no. 1(29), 113–182 (1949). (Russian)

Translation of the first four chapters of Hopf's Ergodentheorie [Ergebnisse der Math., v. 5, no. 2, Springer, Berlin, 1937].

Gurevič, A., and Rohlin, V. On the approximation of non-periodic flows by periodic ones. Doklady Akad. Nauk SSSR (N.S.) **64**, 619–620 (1949). (Russian)

The authors consider the metric space of all measurable measure preserving flows on the unit interval (with the customary identifications modulo sets of measure zero). The distance d between $\{S_t\}$ and $\{R_t\}$ is defined to be the outer measure of the set of those points x for which $R_t x \neq S_t x$ for at least one t, $0 < t \leqq 1$. Let $\mathfrak{P}_p$ be the set of all periodic flows of period p. In the discrete analogue of this situation (involving powers of a single transformation instead of flows) the reviewer proved that, for any nowhere periodic S, $d(S, \mathfrak{P}_p) \leqq 4/p$ [Ann. of Math. (2) **45**, 786–792 (1944); these Rev. **6**, 131]; Rohlin subsequently announced the sharper result $d(S, \mathfrak{P}_p) = 1/p$ [same Doklady (N.S.) **60**, 349–351 (1948); these Rev. **9**, 504]. The purpose of this note is to announce [without proofs] that the same results are valid for flows. The authors assert also that (just as in the discrete case) this approximation theorem implies that the set of weakly mixing flows is a dense G_δ in the complete metric

space of all nowhere periodic flows, whereas the set of strongly mixing flows is a set of the first category.

P. R. Halmos (Chicago, Ill.).

Nakano, Hidegorô. Ergodic theorems in semi-ordered linear spaces. Ann. of Math. (2) **49**, 538–556 (1948).

The mean-ergodic theorem of J. von Neumann and the individual-ergodic theorem of G. D. Birkhoff were originally both theorems about functions of points. They asserted: suppose that in a measure space S, a function-operator $f \to Tf$ is derived from a point-mapping $x \to Mx$ by $(Tf)(x) = f(Mx)$ for all x. Then the averages $f_n = n^{-1}(f + Tf + \cdots + T^{n-1}f)$ converge to a limit f^*. The convergence is "in mean" in the von Neumann theorem, and "for fixed x, for almost all x" in the Birkhoff theorem, and each theorem requires certain restrictions on M and f. Even the original von Neumann proof showed that the mean-ergodic theorem involved points in a nonessential way; in its present-day form, the theorem is about a linear operator in a certain type of Banach space. The norm of the Banach space is required to be uniformly convex or to be related in a certain way to a partial-ordering of the Banach space, but no mention is made of points, point-mapping, or of any abstraction of these notions.

In the Birkhoff theorem, however, the point-mapping plays an essential role. The present paper gives a point-free form of the Birkhoff theorem by abstraction of the relevant features. The functions $f(x)$ on S are replaced by the elements $p, q, \cdots$ of a partially-ordered vector space $\mathfrak{M}$ which is conditionally countably-complete; the points x are replaced by the projections $[p]$ which form a Boolean ring; the linear operator T on $\mathfrak{M}$ is required to satisfy the condition $T([p]q) = [Tp](Tq)$ for all p, q in $\mathfrak{M}$, which abstracts the role of the point-mapping M; finally, in the conclusion of the theorem, the convergence "for almost all x" is replaced by a weak form of order-convergence in $\mathfrak{M}$, namely, $[p]f_n$ should be order-convergent to $[p]f$ for a class of $[p]$ such that every $[q]$ is the union of a subclass of such $[p]$.

The paper gives a theorem about this weak form of order-convergence. Then a "point-free" proof is given for an abstraction of the "cutting-up" lemma in the Birkhoff proof of the ergodic theorem. This leads to an ergodic theorem in $\mathfrak{M}$ provided there are sufficiently many linear functionals on $\mathfrak{M}$, which is shown to be the case whenever $\mathfrak{M}$ is a Banach vector lattice. The final theorem of the paper generalizes the present-day form of the Birkhoff ergodic theorem.

I. Halperin (Kingston, Ont.).

Theory of Probability

Aitken, A. C. A note on the "problème des rencontres." Edinburgh Math. Notes **37**, 9–12 (1949).

A fresh treatment is given of this celebrated problem by means of permanents which differ slightly from those used by E. G. Olds [Bull. Amer. Math. Soc. **44**, 407–413 (1938)] for similar but more complicated problems. The author is mainly interested in the (known) Poisson limit, mean unity, for the probability that x of n permuted elements are not in their own positions, as evidenced by the factorial moment generating function $1+\alpha+\cdots+\alpha^n/n!$. *J. Riordan.*

Epstein, Benjamin. A modified extreme value problem. Ann. Math. Statistics **20**, 99–103 (1949).

Particles are distributed in a region at random in accordance with the Poisson law with mean λ. They vary in magnitude according to a given distribution function $F(x)$ with $F(a)=0$ and $F(b)=1$. Assuming statistical independence, the particles of size not exceeding x have a Poisson distribution with mean $\lambda \cdot F(x)$. Hence it is easy to calculate the probability distributions of the largest, smallest, second-largest, etc. particle in a certain region. *W. Feller.*

Moran, P. A. P. The interpretation of statistical maps. J. Roy. Statist. Soc. Ser. B. **10**, 243–251 (1948).

Suppose that the countries of a map are classified as black and white. The color may be determined either by Bernoulli trials or by a random distribution of fixed numbers of white and black dots. In either case the number of adjacent countries of like color is a random variable. The author shows how its moments can be calculated by purely combinatorial methods. He explicitly calculates the first four moments for the case of a rectangular array of countries. It is shown that the distribution is asymptotically normal.

W. Feller (Ithaca, N. Y.).

Gál, István Sándor. Sur l'ordre de grandeur des fonctions sommables. C. R. Acad. Sci. Paris **228**, 636–638 (1949).

Let $f_n(x) \in L_p(0, 1)$, where $p \geqq 1$ is fixed, and suppose that for $M \geqq M_0$ and all $N \geqq 1$

$$(*) \qquad \int_0^1 |f_{M+1}(x) + \cdots + f_{M+N}(x)|^p dx \leqq cN^h.$$

Then

$$(**) \qquad \sum_{n=1}^{N} f_n(x) = o((N^h(\log N)^{1+\epsilon})^{1/p})$$

for almost all x. This is a special case of a more general theorem in which the interval (0, 1) is replaced by a domain of a Euclidean space, and the right side of the condition (*) is replaced by an arbitrary function $\Phi(M, N, p)$; the inequality (*) is supposed to hold for a certain set P of integers p. Under these conditions the right side of (**) can be replaced by certain functions which depend in a complicated way on P and $\Phi(M, N, p)$. As a special case the relation $\limsup |R_N(x)|(2N \log\log N)^{-\frac{1}{2}} \leqq 1+\epsilon$ for the partial sums of Rademacher functions is derived. [This is the simple part of Khintchine's law of the iterated logarithm.] No proofs are given. *W. Feller* (Ithaca, N. Y.).

Horton, H. Burke, and Smith, R. Tynes, III. A direct method for producing random digits in any number system. Ann. Math. Statistics **20**, 82–90 (1949).

Suppose that the random digits $0, 1, \cdots, n-1$ are produced by means of independent trials so that the outcome k $(0 \leqq k \leqq n-1)$ at the sth trial has probability $p_{k,s}$. Add the digits obtained in s trials and reduce the sum modulo n. Let $\pi_{k,s}$ be the probability that the result is k. Using the technique of nonstationary Markov chains, the authors show that $\pi_{r,s} \to 1/n$ as $s \to \infty$. A similar result is derived for the case of random numbers with fixed but unequal probabilities. The preparation and tests of a set of random digits are discussed in detail. *W. Feller* (Ithaca, N. Y.).

Baticle, Edgar. Sur une loi de probabilité a priori pour l'interprétation des résultats de tirages dans une urne. C. R. Acad. Sci. Paris **228**, 902–904 (1949).

Dumas, Maurice. Interprétation de résultats de tirages exhaustifs. C. R. Acad. Sci. Paris **228**, 904–906 (1949).

The two papers are concerned with the so-called a posteriori probability of the composition of an urn from which a certain specified sample has been drawn. The composition is

considered a random variable and the discussion concerns mainly its a priori distribution. For obvious reasons details of the arguments can not be described adequately here. In a postscript Borel states that he presented the notes as a matter of principle, but that he disagrees with the contents.

W. Feller (Ithaca, N. Y.).

*Lévy, Paul. **Processus Stochastiques et Mouvement Brownien. Suivi d'une note de M. Loève.** Gauthier-Villars, Paris, 1948. 365 pp. 3200 francs.

This is a continuation of the author's earlier book [Théorie de l'Addition des Variables Aléatoires, Gauthier-Villars, Paris, 1937] and contains the results on stochastic processes and Brownian motion mainly obtained by the author during the period 1934–1939. Many of the results of this book have already been published in two papers of the author [Compositio Math. **7**, 283–339 (1939); Amer. J. Math. **62**, 487–550 (1940); these Rev. **1**, 150; **2**, 107], but there are also many new results. The book consists of an introduction and 8 chapters, and is concluded with an appendix by M. Loève.

Chapter I is a brief exposition of two examples of additive, and stationary processes [see chapters V and IV], i.e., Laplace-Gauss processes (Brownian motions) and Poisson processes.

Chapter II gives a general theory of stochastic processes. The author's definition of stochastic process is intuitive rather than measure-theoretic. For example, the Borel field on which the measure (probability) is defined is never explicitly stated. Various kinds of continuity and the notions of differentiation and integration of stochastic processes are introduced. Stochastic differential and integral equations are discussed.

Chapter III is an introduction to the theory of Markov processes with a continuous time parameter. The roles of Chapman and Kolmogoroff's equation and of Kolmogoroff's partial differential equation (equation of the diffusion of probability) are investigated. As an example, the case of Brownian motion is discussed in detail. In this case the equation of diffusion is nothing but the equation of heat conduction.

Chapter IV is devoted to the study of stationary processes, i.e., those stochastic processes $X(t)$ for which $\{X(t_k)\,|\,k=1, \cdots, n\}$ and $\{X(t_k+s)\,|\,k=1, \cdots, n\}$ have the same n-dimensional distribution for any $t_1, \cdots, t_n$ (n is also arbitrary) and for any s. Further, $X(t)$ is called stationary of order p if it is an L^p-process (i.e., if $X(t)$ has moments of order p for each t) and if the condition above is satisfied for $n=p$. The theory of A. Khintchine [Math. Ann. **109**, 604–615 (1934)] plays a fundamental role. The author mentions the recent works of H. Cramér [Ark. Mat. Astr. Fys. **28B**, no. 12 (1942); these Rev. **4**, 13] and M. Loève [Revue Sci. **83**, 297–303 (1945); these Rev. **8**, 38] on harmonic analysis of stationary processes. Among other things it is proved that in order that a mean-square-continuous process $X(t)$ (i.e., the expectation of $|X(s)-X(t)|^2$ tends to 0 as $s\to t$) be stationary of order 2, it is necessary and sufficient that $X(t)$ be of the form $X(t)=\int_{-\infty}^{\infty} e^{it\lambda}\,dU(\lambda)$, where $U(\lambda)$ is an L^2-process for which increments in two disjoint time intervals are orthogonal.

Chapter V is devoted to the study of additive processes (differential processes), i.e., those stochastic processes for which increments in disjoint time intervals are independent. The first half of this chapter is more or less the same as chapter VII of the author's book of 1937. The result of the author on integral representation of infinitely divisible laws is reproduced here. Further, stable laws and Pearson's laws are discussed. This chapter is concluded with discussions on additive processes on circles, spheres and other homogeneous spaces of dimension greater than 1. In particular, results of F. Perrin [Ann. École Norm. Sup. (3) **45**, 1–51 (1928)] on Brownian motion of rotations are discussed in detail.

Chapter VI is a detailed discussion of one-dimensional Brownian motion. Most of the results of this chapter are taken from the author's Compositio paper. Let $X(t)$ be a one-dimensional Brownian motion with $X(0)\equiv 0$. It is shown that $Y_0(t)=|X(t)|$, $M(t)=\max_{0\le s\le t}X(s)$, $Y_1(t)=M(t)-X(t)$, $-m(t)=\max_{0\le s\le t}(-X(t))$, $Y_2(t)=X(t)-m(t)$ have the same distribution for any fixed $t>0$, and also that $Y_0(t)$, $Y_1(t)$, $Y_2(t)$ are even stochastically equivalent, i.e., that $\{Y_j(t_k)\,|\,k=1, \cdots, n\}$ $(j=1, 2, 3)$ have the same n-dimensional distribution for any positive $t_1, \cdots, t_n$ (n also arbitrary). Various arccosine and arcsine laws are proved; for example, for any $t>0$, let T_0 (T_1) be the largest (smallest) zero point of $Y(t)$ which is not larger (not smaller) than t. Then

$$\Pr\{T_0<t_0\}=(2/\pi)\arcsin(t/t_0)^{\frac{1}{2}},$$
$$\Pr\{T_1<t_1\}=(2/\pi)\arccos(t/t_1)^{\frac{1}{2}}.$$

Finally, iterated logarithmic laws and local Lipschitz conditions are discussed.

Chapter VII is a discussion of two-dimensional Brownian motion $Z(t)=(X(t), Y(t))$, where $X(t)$, $Y(t)$ are mutually independent one-dimensional Brownian motions. This chapter overlaps largely with the author's Amer. J. Math. paper. It is proved that in the Brownian motion in 2-space R^2 almost all paths (corresponding to $0\le t<\infty$) constitute an everywhere dense subset of R^2 whose area (2-dimensional Lebesgue measure) is zero, and come back to any neighborhood of any point infinitely many times as $t\to\infty$, while already in 3-space almost all paths of Brownian motion tend to ∞ as $t\to\infty$. It is also shown that we cannot speak of the area (in the classical sense) of the domain D bounded by a path $C(0, t)=\{Z(s)\,|\,0\le s\le t\}$ and a chord connecting two end-points $Z(0)$, $Z(t)$ of $C(0, t)$, but that we can introduce the notion of stochastic area of such a domain D as a stochastic limit of the area of a sequence of polygons which approximate D in a suitable way. This may be considered as a first step in the theory of general stochastic integrals which contains Wiener's stochastic integral [Acta Math. **55**, 117–258 (1930), in particular, pp. 214–234] as a special case. Finally, the invariance under conformal mapping of intrinsic properties of the paths of Brownian motion in R^2 is proved. This fact indicates the close relationship between the theory of analytic or harmonic functions and that of the 2-dimensional Brownian motion.

Chapter VIII is a discussion of Brownian motion with many parameters, i.e., those stochastic processes $\{X(a)\,|\,a\in R^p\}$ $(p\ge 1)$ with the properties: (i) for any $a, b\in R^p$, $X(b)-X(a)$ has a Gaussian distribution with mean 0 and variance $d(a, b)$ (the distance of a and b in R^p), (ii) for any $a_0, a_1, \cdots, a_n\in R^p$, the n-dimensional distribution of $\{X(a_k)-X(a_0)\,|\,k=1, \cdots, n\}$ is Gaussian. The existence of such a process is shown by using a lemma of I. Schoenberg and L. Schwartz on positive definite functions. The continuity, orthogonality and other geometric properties of such processes are discussed. Note that this may be considered as a problem of isometric embedding of a p-dimensional space R^p with $\{d(a, b)\}^{\frac{1}{2}}$ as its metric into a Gauss subspace of a Hilbert space (i.e., a closed linear subspace of a concrete L^2-space such that every

function belonging to it has a Gaussian distribution with mean 0).

The appendix by Loève is an exposition of his theory of stochastic processes, and contains the proofs of those results on harmonic analysis of stationary processes which were recently announced by him in five notes [C. R. Acad. Sci. Paris 220, 295–296, 380–382 (1945); 222, 469–470, 628–630, 942–944 (1946); these Rev. 7, 129, 458]. *S. Kakutani.*

Linnik, Yu. V. On the theory of nonuniform Markov chains. Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 65–94 (1949). (Russian)

Consider an infinite sequence of sequences of chance variables, the nth sequence being $X(1, n), X(2, n), \cdots, X(n, n)$; let $X(h, n)$ take the $k(h, n)$ values

$$a(1, h, n), \cdots, a(k(h, n), h, n).$$

It is assumed that (1) $|a(i, h, n)| < K_0$ for all i, h, n;

$$(2) \qquad [k(h, n)]^{-1} \sum_{i=1}^{k(h,n)} [a(i, h, n) - \xi(h, n)]^2 > c_0,$$

where $\xi(h, n) = [k(h, n)]^{-1} \sum_{i=1}^{k(h,n)} a(i, h, n)$; (3) $2 \leqq k(h, n) \leqq K_1$ for all n and $h \leqq n$; (4) for all n, $X(1, n), \cdots, X(n, n)$ form a simple Markov chain. Define

$$P(i, l, h, n) = P\{X(h, n) = a(l, h, n) \mid X(h-1, n) = a(i, h-1, n)\},$$

$$S(n) = \sum_{i=1}^{n} X(i, n), \quad A(n) = E(S(n)),$$

and $B(n)$ = variance of $S(n)$. The author proves that the distribution of $[B(n)]^{-\frac{1}{2}}[S(n) - A(n)]$ approaches the normal distribution (with mean zero and variance one) if, for some $\epsilon_0 > 0$, $P(i, l, h, n) \geqq n^{\epsilon_0 - 1}$. The theorem is no longer true if we are given only that $P(i, l, h, n) \geqq n^{-1}$. This problem was studied earlier by S. N. Bernstein and N. A. Sapogov, who obtained a weaker result. The author improves on their methods in a long and elaborate proof.

J. Wolfowitz (New York, N. Y.).

Consael, R. Sur une généralisation du processus de Pólya. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 34, 863–876 (1948).

The author considers a process in which the sample functions change in unit jumps, going from n at time t to $n+1$, with probability $\lambda(t)(k+n)dt$, or to $n-1$, with probability $\mu(t)ndt$, at time $t+dt$, independently of previous changes. If $\mu(t) \equiv 0$ the process is essentially the classical Pólya process. Following in detail the work of D. G. Kendall [Ann. Math. Statistics 19, 1–15 (1948); these Rev. 9, 451], who treated the case $K = 0$, the author finds the transition probabilities of the process by solving a partial differential equation in the appropriate generating function. *J. L. Doob.*

Leslie, P. H. On the distribution in time of the births in successive generations. J. Roy. Statist. Soc. Ser. A. 111, 44–53 (1948).

A matrix treatment of the problem of determining the distribution in time of the births in successive generations to a population of arbitrary initial age distribution, subject to given, constant, age-specific fertility and mortality rates. The author indicates that nothing essentially new has been added to the theory as given by Lotka [Ann. Math. Statistics 13, 115–126 (1942); these Rev. 4, 28; and earlier papers], but that the matrix development may have computational advantages. *C. P. Winsor* (Baltimore, Md.).

Weston, J. D. A note on the theory of communication. Philos. Mag. (7) 40, 449–453 (1949).

A message is a real function $f(t)$ of the time t; energy enters the transmission system at a rate proportional to $f(t)^2$. The ideal transmission system cuts out frequencies outside a certain band, and so, by suitable normalization, can be considered as acting to project $f(t)$ on the linear manifold M of functions whose Fourier transforms vanish outside the interval $(-\pi, \pi)$. It is then natural to take as unit signals those whose Fourier transforms are the functions $\{e^{nis}\}$ in $(-\pi, \pi)$ and vanish outside. The unit signals form a complete orthogonal set in M and the received signal can always be expressed as the sum of an orthogonal series in the unit signals. Reception consists in observing the coefficients in this series. *J. L. Doob* (Ithaca, N. Y.).

Benedikt, Elliot T. Errors in second-order measuring instruments. Rev. Sci. Instruments 20, 229–233 (1949).

A quantity $\theta(t)$ is subject to a second order nonhomogeneous linear differential equation in which the coefficients and the term in t alone are random functions. [The statistical hypotheses are not very clearly stated and the concepts of stationarity and ergodicity are confused.] The dispersion of $\theta(t)$ is calculated approximately. *J. L. Doob.*

***Mahdavi Ardebili, Mohammad Hassan. Étude de Certaines Intégrales Multiples de la Théorie des Probabilités Géométriques.** Thesis, University of Geneva, 1940. 54 pp.

Mathematical Statistics

Cansado Maceda, E. On the compound and generalized Poisson distributions. Ann. Math. Statistics 19, 414–416 (1948).

Feller [same Ann. 14, 389–400 (1943); these Rev. 5, 209] has defined (I) compound and (II) generalized Poisson distributions as averages of Poisson distributions over the parameter and as the distribution of a sum of a Poisson number of independent identical random variables, respectively. The author relates the factorial moment generating functions of (I) and (II) to the moment generating functions (ordinary for (I), factorial for (II)) of the other distributions involved. He shows that both classes are closed under convolution and finds a sufficient condition for a distribution to belong to both classes. An infinite number of distributions satisfy the condition, but examples show it is not necessary. *J. W. Tukey* (Princeton, N. J.).

Epstein, Benjamin. Some applications of the Mellin transform in statistics. Ann. Math. Statistics 19, 370–379 (1948).

Simple problems of the distribution of products and quotients are shown to be solvable by Mellin transforms.

J. W. Tukey (Princeton, N. J.).

Kašanin, R. Le coefficient d'approximation moyenne et le coefficient de corrélation. Acad. Serbe Sci. Publ. Inst. Math. 1, 71–87 (1947).

The author develops the by now classical variation of the Chebyshev lemma in which the existence of the $2k$th moment in place of the variance is assumed and discusses some of its consequences. He next turns to an exposition of the approximation of a given set of values by means of a

linear combination of a suitably chosen set of functions of an independent variable on the least squares principle.

C. C. Craig (Ann Arbor, Mich.).

*Fix, Evelyn. **Distributions which lead to linear regressions.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 79–91. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

This paper is concerned with the following problem proposed by R. Frisch. Suppose that $X=a\xi+\alpha$, $Y=b\xi+\beta$, in which ξ, α, and β are mutually independent random variables and a and b are unknown constants. Under what conditions is the regression of Y on X, and also that of X on Y, linear, irrespective of the values of a and b? H. V. Allen gave a partial answer [Statist. Res. Mem. London **2**, 60–68 (1938)] in which it was required, in considering the regression of Y on X, that the first two moments of β, and all the moments of ξ and α, exist. The present author first notes that the statement of the problem implies that the first moment of each of ξ, α, and β exists, and then gives the solution of Frisch's problem in three theorems. Two of these show that necessary and sufficient conditions are that the characteristic functions of the variables ξ and α, if the regression of Y on X is in question, must have one of two given forms, depending on whether a has a definite sign or not. The third theorem further states that if the second moment of either ξ or α exists, then both ξ and α must be normally distributed. *C. C. Craig* (Ann Arbor, Mich.).

Matérn, Bertil. Independence of non-negative quadratic forms in normally correlated variables. Ann. Math. Statistics **20**, 119–120 (1949).

The author proves that, if the covariance of two non-negative quadratic forms in normally distributed variables with zero means vanishes, the two forms are stochastically independent. *C. C. Craig* (Ann Arbor, Mich.).

*Wold, Herman. **Random Normal Deviates. 25,000 Items Compiled from Tract No. XXIV (M. G. Kendall and B. Babington Smith's Tables of Random Sampling Numbers).** Cambridge University Press, 1948. xiii+51 pp. $1.25.

The normal deviates are given to two decimal places. Four different tests of randomness and normality were applied to all blocks of 500, 5,000, and 25,000 numbers. These tests are a sum test, square sum test, range test, and sign run test, for each of which the rejection level was $P=2\%$. One set of 5,000 failed to pass the range test, three sets of 500 failed to pass the sign run test, and one set of 500 failed to pass the square sum test. Finally the agreement of the distribution of P values in each of these tests with the rectangular distribution was tested by the χ^2 method and by a method of Kolmogoroff. [The author wishes to point out two misprints (corrected on an erratum slip in later copies): p. 33, column 7, line 41, read $-.73$ instead of .73; p. 51, column 10, line 12, read -1.09 instead of 1.09.]

H. Chernoff (Chicago, Ill.).

Geary, R. C. Studies in relations between economic time series. J. Roy. Statist. Soc. Ser. B. **10**, 140–158 (1948).

Contributions mainly to confluence analysis, based on the usual assumption that the residuals x''_{it} are non-autocorrelated. The author first points out the close connexion with Hotelling's method of principal components by showing that if there are r linear relations between the p systematic parts x'_{it}, the maximum likelihood estimates of the x'_{it} are linear forms in the first $p-r$ principal components. Next follows a simple deduction of the large-sample test for r which G. Tintner [Ann. Math. Statistics **16**, 304–308 (1945); these Rev. **7**, 132] has derived from a result of P. L. Hsu. The chief result is a new method for confluence analysis: the x'_{it} are estimated by expansion in terms of orthogonal functions; the method includes a χ^2 test for r, and a large-sample variance formula for the ratio between two coefficients in the relation under estimation. An illustration deals with Kutznets's and Barger's economic series for the U. S. A., 1921–38; a remarkable feature is that the intercorrelations between the six series considered are not much affected by the removal of orthogonal polynomials up to degree 7.

H. Wold (Uppsala).

Anderson, T. W. The asymptotic distributions of the roots of certain determinantal equations. J. Roy. Statist. Soc. Ser. B. **10**, 132–139 (1948).

The author investigates the asymptotic distribution of the roots of a determinantal equation involving a set of sample regression coefficients, and also the asymptotic distribution of the roots of a determinantal equation involving a noncentral sample covariance matrix. Let X_{it} $(i=1,\cdots,p;t=1,\cdots,N)$ be normally distributed with mean $EX_{it}=\sum_{\alpha=1}^{m}\varphi_{i\alpha}\xi_{\alpha t}$, covariance $E(X_{it}-EX_{it})(X_{jt}-EX_{jt})=\sigma_{ij}$, and independently of X_{ks} for $s\neq t$; further suppose that $m_{\alpha\beta}=N^{-1}\sum_{t=1}^{N}\xi_{\alpha t}\xi_{\beta t}$ is an element of a matrix of rank m, and $\lim_{N\to\infty} m_{\alpha\beta}=M_{\alpha\beta}$ is an element of a matrix of rank m. Let $\|m^{\alpha\beta}\|=\|m_{\alpha\beta}\|^{-1}$, $\alpha,\beta=1,\cdots,m$, and $\|n_{\alpha\beta}\|=\|m^{\alpha\beta}\|^{-1}$, $\alpha,\beta=1,\cdots,q\ (\leq m)$. Let $\rho_1^2,\cdots,\rho_h^2,\rho_{h+1}^2=0$ be the distinct roots (numbered in descending order of size) of $|\sum_{\alpha,\beta=1}^{q}\varphi_{i\alpha}\varphi_{j\beta}n_{\alpha\beta}-\rho^2\sigma_{ij}|=0$ with multiplicities $r_1,\cdots,r_h,r_{h+1}=r$, respectively. Let $\mu_1,\cdots,\mu_p$ be the roots (numbered in nonascending order) of

$$\left|\sum_{\alpha,\beta=1}^{q} f_{i\alpha}f_{j\beta}n_{\alpha\beta}-\mu\sigma_{ij}\right|=0,$$

where $f_{i\alpha}$ is the sample regression of X_{it} on $\xi_{\alpha t}$. Under these conditions the asymptotic distribution of

$$\Theta_i=\tfrac{1}{2}N^{\frac{1}{2}}(\mu_i-\rho_k^2)\rho_k^{-1},\quad i=r_1+\cdots+r_{k-1}+1,\cdots,r_1+\cdots+r_k,\ k=1,\cdots,h,$$

$$\Theta_i=N\mu_i,\quad i=p-r+1,\cdots,l_1=\min(p,q),$$

is determined. Now let $\Lambda=N(\mu_{p-r+1}+\cdots+\mu_p)$ be the test of the hypothesis that the number of linear relations is r against the alternatives that the number is less than r; then the asymptotic distribution of Λ is the χ^2 distribution with $r(q-p+r)$ degrees of freedom. The remaining two theorems involve a noncentral sample covariance matrix and their precise statements are lengthy. In all the theorems the condition of normality may be relaxed. The results were prompted by the economic models of R. S. Geary [see the preceding review], and G. Tintner [Ann. Math. Statistics **16**, 304–308 (1945); these Rev. **7**, 132]. Both models are discussed. *L. A. Aroian* (New York, N. Y.).

Narain, R. D. Frequency distribution of χ^2-constituents under a linear constraint. Proc. Benares Math. Soc. (N.S.) **8**, no. 1, 33–39 (1946).

Let y_i be distributed independently as

$$(2\pi)^{-\frac{1}{2}}\sigma^{-1}\exp\{-\tfrac{1}{2}\sigma^{-2}(y_i-bc_i)^2\},$$

where $b=\sum c_iy_i/\sum c_i^2$, $i=1,2,\cdots,n$, not all c's zero. Denote the residuals Y_i by y_i-bc_i, subject to the condition

$\sum_{i=1}^{n} c_i Y_i = 0$. The author finds the joint probability function of any $p \leqq n-1$ of the residuals $Y_1, Y_2, \cdots, Y_p$, thus generalizing a result of Irwin [J. Roy. Statist. Soc. (2) **92**, 580–584 (1929)]. *L. A. Aroian* (New York, N. Y.).

Madow, William G. On the limiting distributions of estimates based on samples from finite universes. Ann. Math. Statistics **19**, 535–545 (1948).

The paper deals with the limiting distribution of the arithmetic means for samples selected from a sequence of finite universes and without replacement. It is shown that the limiting distribution under general conditions is normal. The proof is based on the fact that the moments of the means tend to the moments of the normal distribution. It seems to the reviewer that the formula in corollary 1 contains a few misprints (e.g., the letter m is not explained). *K. R. Buch* (Copenhagen).

Haldane, J. B. S. The precision of observed values of small frequencies. Biometrika **35**, 297–300 (1948).

The cube root is suggested as a normalizing transformation in the estimation of rare events. The derivation is based on inverse probability. *C. P. Winsor* (Baltimore, Md.).

Pearson, E. S. Note on Professor Haldane's paper regarding the treatment of rare events. Biometrika **35**, 301–303 (1948).

A discussion of Haldane's paper [see the preceding review] with alternative treatments avoiding the use of inverse probability. *C. P. Winsor* (Baltimore, Md.).

Hoeffding, Wassily. A non-parametric test of independence. Ann. Math. Statistics **19**, 546–557 (1948).

Let Ω' be the class of continuous distribution functions and Ω'' that with continuous joint and marginal probability densities. A rank test for independence of two random variables, consistent with respect to Ω'', is given. The asymptotic behaviour of the test statistic D is studied; general results of the author's previous paper [same vol., 293–325 (1948); these Rev. **10**, 134] are applied. It is shown that there do not exist rank tests of independence which are unbiased on any significance level with respect to Ω' or Ω''. *M. Loève* (Berkeley, Calif.).

Walsh, John E. Some significance tests for the median which are valid under very general conditions. Ann. Math. Statistics **20**, 64–81 (1949).

Consider a set of n observations, one from each of n populations; let these n observations ordered in increasing magnitude be denoted by $x_1, \cdots, x_n$. It is assumed that each distribution (i) is continuous, (ii) has the median ϕ, and (iii) is symmetric about ϕ. The author uses the ordered observations to give explicitly significance tests of the hypothesis $H: \phi = \phi_0$ for $n = 4, \cdots, 15$. For example, $\frac{1}{2}(x_1 + x_2) > \phi_0$ defines a critical region (of a one-sided test) of size 6.2% for $n = 5$. If the n distributions are normal with common variance, the power of such a test compares favorably with the corresponding t-test. In some general theorems the author proves that the sizes of the critical regions are independent of the distributions, and he indicates how other tests may be constructed. *T. W. Anderson.*

Seth, G. R. On the variance of estimates. Ann. Math. Statistics **20**, 1–27 (1949).

The author has extended and improved a number of results on the lower bound of the variance of unbiased estimates. Let $X_1, X_2, \cdots$ be a sequence of absolutely continuous random variables with probability density $p_M = p_M(x_1, \cdots, x_M; \theta)$. (The author indicates the minor changes necessary when $X_1, X_2, \cdots$ are discrete random variables.) Let $\Phi_\alpha(x_1, \cdots, x_\alpha)$ $(\alpha = 1, 2, \cdots)$ be an infinite sequence of functions defined for all observable sequences $x_1, x_2, \cdots$ such that each function takes only the values zero or one, and let n be the first value of α for which $\Phi_\alpha(x_1, \cdots, x_\alpha) = 1$. Let $\theta^* = \theta^*(x_1, \cdots, x_n)$ be an unbiased estimate of the real valued function $\gamma(\theta)$ of the parameter θ, $\phi_i(M) = p_M^{-1} \partial^i p_M / \partial \theta^i$, $\lambda_{ij} = E[\phi_i(n) \cdot \phi_j(n)]$ $(i, j = 1, \cdots, m)$, and $\|{}_{(m)}\lambda^{ij}\| = \|\lambda_{ij}\|^{-1}$. Then, subject to certain regularity conditions too complicated to state here, it is shown that

$$(*) \qquad \sigma^2(\theta^*) \geqq \sum_{i,j=1}^{m} {}_{(m)}\lambda^{ij} \frac{d^i \gamma(\theta)}{d\theta^i} \frac{d^j \gamma(\theta)}{d\theta^j}.$$

This generalizes a result of Bhattacharyya [Sankhyā **8**, 1–14 (1946); these Rev. **8**, 524] for a fixed sample size, and also a result of Wolfowitz [same Ann. **18**, 215–230 (1947); these Rev. **9**, 49] for the sequential case. A necessary and sufficient condition is obtained that for some $m > 1$ the lower bound in (*) is greater than for $m = 1$. Two examples of a Wald sequential procedure are given for which the bound in (*) with $m = 2$ is greater than the Wolfowitz lower bound corresponding to $m = 1$. Necessary and sufficient conditions are given that for a fixed sample size the lower bound in (*) is actually achieved; when $m = 1$, these conditions reduce to the Cramér conditions for an "efficient" estimate [H. Cramér, Mathematical Methods of Statistics, Princeton University Press, 1946, p. 480; these Rev. **8**, 39]. Much of the foregoing has also been extended by the author to the multiparameter case. *E. Paulson* (Seattle, Wash.).

Noether, Gottfried E. Confidence limits in the non-parametric case. J. Amer. Statist. Assoc. **44**, 89–100 (1949).

The purpose of this article is to give a survey of certain methods available for finding confidence limits when nothing is assumed about the population from which a sample has been drawn except possibly that it has a continuous distribution. The following three cases are treated: a confidence band for the unknown cumulative distribution function, a confidence interval for the proportion of a population for which the variate is smaller (or larger) than a given value, and confidence intervals for quantiles. It is the purpose of this paper to state known results without referring to any mathematical proofs. *From the author's summary.*

Nair, K. R., and Rao, C. R. Confounding in asymmetrical factorial experiments. J. Roy. Statist. Soc. Ser. B. **10**, 109–131 (1948).

The authors consider the following designs for experiments with m factors at levels $s_1, \cdots, s_m$ in b blocks of k each. (1) Every treatment combination occurs the same number r of times. (2) The treatments $(i_1, \cdots, i_m)$ and $(j_1, \cdots, j_m)$ occur together in $\lambda(k_1, \cdots, k_m)$ blocks, where $k_t = 0$ if $i_t = j_t$ and $k_t = 1$ if $i_t \neq j_t$ and $(x_1, \cdots, x_m)$ denotes an experiment with the jth factor $(j = 1, \cdots, m)$ at level x_j. These designs are termed balanced confounded designs and have the following properties. (a) The differences of the maximum-likelihood estimates of the main effects and interactions are independently distributed. (b) The loss of information (1−efficiency of maximum likelihood estimate) on every component of a particular main effect or interaction is the same. (c) The upper limit of the total loss of information on all the degrees of freedom of the main effects and inter-

actions of any set of $n>1$ out of the m factors is one less than $1/k$ times the number of treatments that may be formed from these factors.

The authors give completely the analysis of variance of balanced confounded designs for two factors. They then define optimum designs of two factors as designs for which there is no loss of information on one factor while the loss of information on the other factor takes its minimum value. Such designs can be obtained for particular values of r. The authors give construction methods for various values of the parameters r, k, b, λ_{01}, λ_{10}, λ_{11} and exemplify them in particular cases. The authors also treat experiments on 3 factors when the number of levels is equal for two of the factors and of 4 factors when the number of levels is equal for the first two and the last two factors.

H. B. Mann (Columbus, Ohio).

Pearce, S. C. Randomized blocks with interchanged and substituted plots. J. Roy. Statist. Soc. Ser. B. 10, 252–256 (1948).

The author describes the analysis of randomized block designs with the following modifications. (1) When an error has occurred in the planting so that say variety 1 has been planted twice in the first block and not at all in the second, while variety 2 occurs twice in the second block and not at all in the first. (2) When for some reason not enough material was available to plant variety 1 in all blocks. Thus variety one occurs in only L of K blocks. Instead of treating the design as a design with missing plots it will increase the efficiency somewhat to substitute another variety, say variety 2, in the remaining $K-L$ blocks for variety 1. For both these cases the author gives the least square estimates of the varietal effects and their differences and describes the proper analysis of variance tests (that is to say the F tests derived from the likelihood ratio principle).

H. B. Mann (Columbus, Ohio).

Mathematical Economics

***Samuelson, Paul Anthony. Foundations of Economic Analysis.** Harvard University Press, Cambridge, Mass., 1947. xii+447 pp. $7.50.

The author's main thesis is that there exist formally identical meaningful theorems in the most varied fields of economic analysis. There is thus a dual purpose: to demonstrate the existence of this unity of method and to deduce meaningful theorems. By a meaningful theorem is meant simply one which is capable of contradiction by the facts. Meaningful theorems are deduced almost wholly from two kinds of hypotheses. The first of these derives from the fact that in many instances the conditions of equilibrium are equivalent to the maximization of some variable. The secondary conditions for an extremum may thus be employed to impose restrictions (generally qualitative) on the variables. Part I is concerned with comparative statics and the theory of maximizing behavior. Chapters 2 and 3 develop the general method along now well-known lines wherein the behavior of an equilibrium system is defined in terms of a given set of functional relations and initial conditions. But nowhere else is the general method developed with the mathematical skill and completeness shown here. The analysis is extended to include the treatment of finite, as well as infinitesimal, changes. The purely mathematical aspect of the problem is supplemented by an appendix on the theory of maxima and quadratic forms which covers most of the theorems useful for economic analysis. Minor objections to the treatment of the theory of extrema may be raised, such as the lack of integration between the terminology and treatment in the text, footnotes, and appendix.

The balance of part I is devoted to detailed analyses of special topics such as the theory of costs and production [chapter 4], consumer behavior and related topics [chapters 5, 6, and 7] and a long discussion of welfare economics wherein the development of the subject is summarized and the various assumptions are analyzed. This limited range of topics is extensively analyzed and the bulk of the false generalizations and misconceptions is eliminated. As an example, one result from the theory of consumer's behavior may be cited. It is shown that

$$\sum_1^n p_i\Delta x_i \leqq 0 \rightarrow \sum_1^n (p_i+\Delta p_i)\Delta x_i < 0.$$

This single expression essentially exhausts the entire empirical content of the theory of consumer's choice and furthermore provides the basis for the theory of index numbers. The theoretical discussion of the latter [chapter 6] is clearly the best that has yet appeared in the literature.

Part II is devoted to the derivation of meaningful theorems in comparative statics from the conditions of dynamic stability. This relation between dynamics and statics is termed the correspondence principle, by which is meant that the equilibrium positions reached under comparative statics have no meaning unless the corresponding dynamic system is stable. It may be shown that certain solutions of a static system are not consistent with the stability of an explicit dynamic model. Hence specified properties of one system will permit the derivation of information concerning the other. The method is especially applicable to the interactions between economic units to which the assumption of maximizing behavior is not always applicable.

A dynamic system is said to be one which determines the time behavior of all included variables starting from arbitrary initial conditions. Given n variables in time, the n functional equations of the general form

$$F^i[x_1{}_{-\infty}^{\ t}(\tau), \cdots, x_n{}_{-\infty}^{\ t}(\tau), t]=0$$

$(i=1, \cdots, n)$ express such a system. While a variety of different forms may be employed, the author relies on finite difference equations. A compact and general appendix on difference equations is provided. Equilibrium solutions of the system are given by the set of constants which satisfy the equations of the system identically. The equilibrium solution is said to possess perfect stability of the first kind if $\lim_{t\to\infty} x_i(t)=x_i^0$ independently of the initial conditions. If the equilibrium is stable for small displacements it is called stability of the first kind in the small. For the most part the author is concerned with first order stability of the first kind in the small. This condition prevails when, in a sufficiently small neighborhood of a point, the linear terms in the expansion of the system of functional equations yield a perfectly stable system. Within the domain of the possible cases that might arise only a limited number are considered and these are selected primarily on the basis of mathematical tractability. No economic criteria are offered for selection, and in spite of the presence of economic examples the economic content of these models is distressingly low.

In chapter 11, which in a sense is an independent essay, a number of aspects of dynamic theory are considered. A classification of economic systems, based in part on Frisch, and involving combinations of static, dynamic, and stochastical on the one hand, and stationary, historical, and causal on the other, is given along with a discussion of the meaning of moving equilibrium. There follows a cursory discussion of various models for business cycle analysis.

M. P. Stoltz (Providence, R. I.).

Tintner, Gerhard. Homogeneous systems in mathematical economics. Econometrica 16, 273–294 (1948).

This is an investigation of the conditions necessary for homogeneity of zero degree of the demand and supply functions in prices and incomes under various economic situations. The main mathematical results on which the conclusions are based may be expressed as follows. Consider 3 column vectors

$$x=\{x_1, \cdots, x_p\},$$
$$x^*=\{x_{p+1}, \cdots, x_r\}, \qquad p\leqq r,$$
$$x^{**}=\{x_{r+1}, \cdots, x_s\}, \qquad r<s,$$

and functions

$$g=g(x, x^*, x^{**}),$$
$$h^{(k)}=h^{(k)}(x, x^*, x^{**}), \qquad k=1, \cdots, n,$$

possessing first and second derivatives; g is to be maximized with respect to $\{x\}$ and $\{x^*\}$ subject to $h^{(k)}=0$. It is now shown that (1) if the function g and the constraints $h^{(k)}$ are all either independent of $\{x^{**}\}$ or homogeneous of some arbitrary degrees in the same variables, then the solutions $x_1, \cdots, x_p$ are homogeneous of zero degree in these variables, and (2) if the g and the $h^{(k)}$ are all either independent of the components of x^* and x^{**} or homogeneous of some arbitrary degrees in the same variables, then the solutions $x_1, \cdots, x_p$ are homogeneous of zero degree in the components of x^{**}.

These results are now applied to the theory of consumer's choice, production with several transformation functions, monopoly and monopsony, monopolistic and monopsonistic competition, various special cases of oligopoly and oligopsony, bilateral monopoly, and the formation of anticipations. The conditions under which homogeneity of zero degree of the demand and supply functions exists are stated for each of these cases. *M. P. Stoltz* (Providence, R. I.).

***Dresch, Francis W. Continuous index numbers and quantitative study of the general economy.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 203–221. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

The author shows how it would be possible by using index formulas of Divisia's continuous type to define price indices and volume indices for different classes of goods in such a way that they can be used in the classical theory of prices and production (in a static model) instead of prices and quantities of individual goods. In an example describing a dynamic model he has to introduce also quantities which measure the variances in the rate of change of the logarithms of individual prices before he is able to transform the theory in such a way that he can use indices for whole groups of goods instead of prices and quantities for individual goods. [There are confusing printing errors in formulas (88), (89), (90), (94), (96), and (98): in (88) there is q_i instead of g_i; in (89) and (90) $v_k{}^i\dot{\gamma}_k{}^i$ instead of $n_k{}^i\dot{\gamma}_k{}^i$; in (94) p_i instead of $\dot{p}_i$; in (96) $(\dot{p}_i)^i$ instead of $(\dot{p}_i)^2$; and in (98) $(P_s\eta_s)^2$ instead of $(\dot{P}_s\eta_s)^2$; in the last sentence after (88) there are $\dot{\gamma}_k{}^i$ and $\dot{\gamma}_k{}^i$ instead of $\dot{\gamma}_k{}^i$ and $\gamma_k{}^i$.] The assumption that the normal amount of stock B_i required for satisfactory operations is a constant in time seems to the reviewer to be a rough approximation. *L. Törnqvist* (Helsingfors).

Mathematical Biology

***Dahlberg, Gunnar. Mathematical Methods for Population Genetics.** S. Karger, Basle; Interscience Publishers, Inc., New York, N. Y., 1948. viii+182 pp. $4.50.

A detailed analysis of genetical and eugenical problems in human populations. First the several forms of inheritance are discussed with special attention to selection and mutations. Then the effects of consanguineous marriages are studied in their dependence on population size. There follow several chapters on assortative mating and the importance of isolates within a population. The numerical analysis is emphasized. In the preface the author states that "in order for the reader to accustom himself in successive steps to the methods of calculation used here, the presentation has been given a very elementary form, especially in the first part of the book which to mathematically trained readers must seem irritating and tedious." This describes the purpose of the book, but does not do justice to the elegance of presentation. [A German edition of the book was published as Supplement no. 148 to Acta Medica Scandinavica, 1943.]

W. Feller (Ithaca, N. Y.).

Levene, Howard. On a matching problem arising in genetics. Ann. Math. Statistics 20, 91–94 (1949).

In a population breeding at random the probabilities of homozygotes and heterozygotes in a system of multiple alleles are calculable from the various gene proportions. The author obtains the distribution, exact and asymptotic, of the number of homozygotes in a sample, given the numbers of the various alleles in the sample. He also considers the effects of misclassification of genotypes in the sample.

C. P. Winsor (Baltimore, Md.).

Fisher, R. A. The theory of linkage in polysomic inheritance. Philos. Trans. Roy. Soc. London. Ser. B. 233, 55–87 (1947).

A treatment of the mathematical problems of polyploid inheritance. These include the enumeration and classification of possible genotypes, the enumeration of modes of gamete formation, and the statistical problems involved in the estimation of the frequencies of the various modes of gamete formation. *C. P. Winsor* (Baltimore, Md.).

Castoldi, Luigi. Attorno a un problema di biologia matematica: il decorso di una infezione per inoculazione di germi patogeni. Atti Accad. Ligure 4 (1947), 75–92 (1948).

The author writes $dN/dt=kN-M$ as the rate of increase of pathogenic germs within the organism, with M representing a destructive capacity of the organism. Cavalli and Magni [Bollettino della Società Medico-Chirurgica di Pavia, fasc. 4–5, 1942] set $M=hN$, where h is constant, but the author proposes to make

$$M(t)=M_0+\int_0^t N(\tau)f(t-\tau)d\tau.$$

Then supposing $f=\lambda=$constant, he obtains a single second order equation in N and devotes most of the paper to an examination of the dependence upon K and λ.

A. S. Householder (Oak Ridge, Tenn.).

TOPOLOGY

Komatu, Atuo. On the weak topology of an infinite product space. J. Math. Soc. Japan 1, 58–62 (1948).

A topology is here taken to be an operator $E \to E^t$ such that $0^t = 0$ (0 is the void set). It is monotone (M) if $E \subset F$ implies $E^t \subset F^t$, and it is additive (A) if $E^t \cup F^t = (E \cup F)^t$. A topology t for a Cartesian product $R = \Phi R_x\ (x \in X)$ may have the property (1) $E^{tx} \subset E^{xt}$, where, for any $F \subset R$, F^x is the image of F under the projection on R_x. The author constructs a weakest topology t in R for which (1) and (M) hold and also a weakest topology for which (1) and (A) hold. The neighborhood definitions of these topologies are also given. *R. Arens* (Los Angeles, Calif.).

Doss, Raouf. On uniform spaces with a unique structure. Amer. J. Math. 71, 19–23 (1949).

It is shown that in order that a completely regular space E have a unique uniform structure it is necessary and sufficient that, for every real-valued continuous function f defined on E, either the set $[f \geqq 1]$ or the set $[f \leqq 0]$ be compact. It is remarked that these conditions imply that E is compact if it is metric, by Dieudonné's theorem that a metric space is complete in some uniform structure [C. R. Acad. Sci. Paris 209, 666–668 (1939); these Rev. 1, 108]. The result stated above rests on the lemma that if F is a filter without limit point in E, then there is a uniform structure in which F is Cauchy. *R. Arens.*

Colmez, J. Sur divers problèmes concernant les espaces topologiques. Les espaces à écarts—problème de Wiener sur les transformations continues. Portugaliae Math. 6, 119–244 (1947).

There are treated at length two distinct topics in the theory of "spaces" with a closure operation subject to no restriction. The first part deals with écartized spaces, which present a generalization of metric spaces presumably more intuitive but equivalent to the uniform structures of A. Weil. An écart is a function $\rho(x, y)$ of two variables in the space with values in an ordered directed set S having a least element and occasionally modified by axioms of symmetry and regularity (cf. triangle inequality). This is a slight generalization of ideas introduced into the literature independently in articles of M. Fréchet [C. R. Acad. Sci. Paris 221, 337–340 (1945); Portugaliae Math. 5, 121–131 (1946); these Rev. 7, 215; 8, 48] and G. K. Kalisch [Bull. Amer. Math. Soc. 52, 936–939 (1946); these Rev. 8, 166]. Kalisch required S to be also a vector space, and could still obtain all uniform structures, while Fréchet required S to be totally ordered. The latter condition was shown by R. Doss to rule out many nonmetric uniform structures [Bull. Sci. Math. (2) 71, 110–122 (1947); these Rev. 9, 605]; the argument is that S has no strictly decreasing co-initial sequence if and only if the space is nonmetric, but then all G_δ-sets are open, leading to what might be called uniformly zero-dimensional spaces.

The second part deals with the "problem of Wiener" which requires the construction of all topologies making a preassigned group G of permutations of an infinite set X into homeomorphisms [N. Wiener, Bull. Soc. Math. France 50, 119–134 (1922)]. Consideration of the lattice of solution topologies shows that there are always nondiscrete solutions satisfying the closure axiom $A \subset A^-$, but sometimes there are no Hausdorff nondiscrete solutions. The solution is never unique. Various methods for constructing solutions satisfying various axiomatic restrictions on closure, feasible when G satisfies suitable conditions, are developed. There is given preparatory material on directed sets, filters, compactness, etc., as well as a 6 page summary. *R. Arens.*

Sierpinski, Waclaw. Remarque sur les espaces topologiques. Revista Ci., Lima 50, 193–196 (1948).

Let A and B be T_1-spaces, and let f, g be continuous mappings of A to B which agree on a dense subset of A. It is pointed out that while this implies $f = g$ when B is a Hausdorff space, it does not imply $f = g$ in general. An example is given. *R. Arens* (Los Angeles, Calif.).

Cotlar, Mischa. An extension of Rolle's theorem for continuous transformations of the plane. Math. Notae 8, 79–84 (1948). (Spanish)

Let $u(x, y)$, $v(x, y)$ be continuous real-valued functions defined for all points (x, y) in the closure of a plane domain D. For $p = (x, y)$, let $f(p)$ be the point (u, v), and $E = f(D)$. The point p of D is called a strong extremum (with respect to f) if there is an analytic simple closed curve γ containing $f(p)$ and such that E lies in γ plus one of its two complementary domains. The theorem corresponding to the title is as follows. Let $w = f(z) = u(x, y) + iv(x, y)$ be a continuous function in a domain D, and suppose the Jacobian $J(z) = u_x v_y - u_y v_x$ exists at every interior point of D except for points of some reducible set. If $u(x, y)$ is constant over the boundary of D then the set of zeros of $J(z)$ contains an infinite irreducible set whose closure is in D. There are two other related theorems.

Several questions are raised. One of them is as follows. If G is a simply connected plane domain then does the boundary of G contain a set K having the power of the continuum such that for $p \in K$, p is accessible by circles (that is, there is a circle γ through p whose interior is in G)? The existence of real continuous functions with no derivative suggests a simple construction for a domain D bounded by a simple closed curve F such that only a countable number of points of F are accessible by circles. *J. H. Roberts.*

Dubois-Violette, Mme. Pierre-Louis. Sur les réseaux de courbes couvrant une surface de genre p. C. R. Acad. Sci. Paris 228, 896–898 (1949).

The author studies families (nets) of curves with a finite number of singularities on a closed orientable surface of genus p. The net is orientable if all the curves can be oriented coherently. Theorem: the net is orientable if and only if every closed curve has an even number of contacts with the net (or can be ϵ-deformed into such a curve). In particular, the index of every singular point must be integral. The net is called semi-orientable if it can be oriented except for a set of measure 0. Examples of semi-orientable, but not orientable, and of not semi-orientable nets are given.

H. Samelson (Ann Arbor, Mich.).

Reeb, Georges. Sur les trajectoires fermées de certains champs de vecteurs. C. R. Acad. Sci. Paris 228, 1097–1098 (1949).

Let E_μ be a family of vector fields, with parameter μ, in a manifold V_n. Suppose the field E_0, for $\mu = 0$, has no singularities, and its integral curves are circles, which fiber V_n, with base space V_{n-1} and projection f. Define a vector field $\bar{E}$ on V_{n-1}, by $\bar{E}(y) = \int_{S_y} E' dt$, S_y the fiber over y, and

$E'(x)=\partial E_\mu(x)/\partial\mu|_{\mu=0}$. The inverse image under f of an $(n-1)$-cell B^{n-1} is homeomorphic to $B^{n-1}\times S^1$. The family E_μ, for sufficiently small μ, defines a mapping S of the set of points sufficiently close to $x_1\varepsilon B^{n-1}$ into B^{n-1}, "by going around the trajectories." The author states several theorems connecting the singularities of the field $\bar{E}$, and the existence of closed trajectories under E_μ for small μ, and the characteristics of V_n and V_{n-1}. These are generalizations of classical results concerning the differential equation $x''+x=\mu f(x,x')$.

H. Samelson (Ann Arbor, Mich.).

Reeb, Georges. Sur les solutions périodiques de certains systèmes différentiels canoniques. C. R. Acad. Sci. Paris **228**, 1196–1198 (1949).

The author gives an application of the theorems developed in an earlier note [see the preceding review] to the case where the vector fields are derived from the differential equations of a canonical Hamiltonian system, depending on a parameter μ, and he obtains theorems concerning closed trajectories. *H. Samelson* (Ann Arbor, Mich.).

Thom, René. Sur une partition en cellules associée à une fonction sur une variété. C. R. Acad. Sci. Paris **228**, 973–975 (1949).

Let f be a twice differentiable function, with a finite number of nondegenerate singularities, on the compact n-manifold V_n. One can introduce a Riemannian metric in V_n, such that grad $f\neq 0$, except at the singular points; each trajectory of the flow, associated with this vector field, goes from a singular point to a higher (larger f) singular point. The trajectories starting at a singular point of type p form an open p-cell, and these cells form a cellular decomposition of V_n, which, however, is not necessarily a cell complex. Denoting the p-skeleton of the cell decomposition by K^p, the number of p-cells by i_p, and the pth Betti number by β_p, one finds, by deformation and retraction, $\beta_p(K^p)\geqq\beta_p(V_n)$, $\beta_p(K^{p+1})\geqq\beta_p(V_n)$, and $i_p\geqq\beta_p(V_n)$ (the Morse inequality). Applications, concerning the fundamental group, etc., are indicated. *H. Samelson* (Ann Arbor, Mich.).

Fenchel, W. Estensioni di gruppi discontinui e trasformazioni periodiche delle superficie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 326–329 (1948).

A geometrical automorphism of the fundamental group F of a surface S is one which is induced by a homeomorphism of S. The author states the theorem: given a discontinuous group F of hyperbolic motions of the unit circle (defining a surface S) and a geometrical automorphism α of F of order n (i.e., α^n is an inner automorphism of F), it is possible to vary the moduli of F so as to obtain a group F' of the same type for which α can be realized by a hyperbolic motion a; adjoining a to F' one obtains a group isomorphic to the group generated by α and the inner automorphisms of F. The proof uses the fact that the manifold of the moduli has a Euclidean space for covering space, and P. A. Smith's fixed point theorem. Several applications are given.

H. Samelson (Ann Arbor, Mich.).

Fáry, István. Sur les groupes d'homéomorphismes également continus du plan. C. R. Acad. Sci. Paris **228**, 534–536 (1949).

Let G be a group of homeomorphisms of the plane, where G has a topology given by means of the metric of a sphere on which the plane has been stereographically projected. The following theorem is announced. If the transformations of G are equicontinuous and if each transformation of G possesses a square root, then G is strictly isomorphic to (i.e., is conjugate to) a subgroup of the Euclidean or hyperbolic group. *D. Montgomery* (Princeton, N. J.).

Fáry, István. Sur la dimension des groupes d'homéomorphismes également continus du plan. C. R. Acad. Sci. Paris **228**, 801–803 (1949).

This paper contains results related to those of the preceding review. It is concerned with groups of transformations of the plane which are equicontinuous and of positive dimension. It is shown that if G is a group of equicontinuous transformations of the plane then dim $G\leqq 3$ and if dim $G=3$ then G is strictly isomorphic to the Euclidean or hyperbolic group. There are similar results when dim $G=1$ or 2.

D. Montgomery (Princeton, N. J.).

Wu, Wen-tsün. Sur la structure presque complexe d'une variété différentiable réelle. C. R. Acad. Sci. Paris **228**, 972–973 (1949).

Following Pontrjagin and Chern, certain cocycles are defined in the space $G_{m,n}$ of m-planes in $(m+n)$-space in the real and complex cases. As is well-known, the tangent bundle of an oriented differentiable m-manifold M may be studied by means of a certain mapping of M into $G_{m,n}$. Using this, certain relations between the characteristic classes of M are given which are stated to be necessary for M to admit an almost analytic structure. *H. Whitney.*

Bassi, Achille. Sopra l'indipendenza di alcuni invarianti topologici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 235–238 (1948).

The author shows, with simple examples, that the combinational, category-type invariants μ_0, μ_1, μ_2 of complexes, defined by him [Ann. Mat. Pura Appl. (4) **16**, 275–297 (1937)], are independent of the homology groups. He remarks also that the invariant μ_0, and the "irregularities" $\mu_0-\mu_1$, $\mu_0-\mu_2$, are independent of the homology ring. The connection with the homotopy groups is still an open problem. *H. Samelson* (Ann Arbor, Mich.).

Liao, S. D. Some theorems on the dimension of fibre spaces. Amer. J. Math. **71**, 231–240 (1949).

"Fibre space" is understood in the sense of Hurewicz and Steenrod [Proc. Nat. Acad. Sci. U. S. A. **27**, 60–64 (1941); these Rev. **2**, 323]; the base space B is assumed to be a polyhedron and the fibre space X separable metric. The projection of X on B is denoted by π. If X is a finite-dimensional compactum and B has homogeneous dimension then $\dim X=\dim B+\max_{b\varepsilon B}\{\dim \pi^{-1}(b)\}$. If X is an irreducibly closed polyhedron the fibres $\pi^{-1}(b)$ must all have the same dimension. Some relationships between dim X, dim B, and $\{\dim \pi^{-1}(b)\}$ are established under less restrictive hypotheses.

R. H. Fox (Princeton, N. J.).

Hirsch, Guy. Un isomorphisme attaché aux structures fibrées. C. R. Acad. Sci. Paris **227**, 1328–1330 (1948).

Let $\tilde{M}$ be a fibre bundle over M with projection P and fibre F. Using cohomology groups with rational coefficients, the author defines, for each dimension p, a characteristic isomorphism of a factor group of a subgroup of the cohomology group $H^p(F)$ onto a group similarly related to $H^{p+1}(M)$. It is stated that one of these, suitably interpreted, is the characteristic cocycle of the bundle. [This could only

be so if the coefficients of the latter (in $\pi_\lambda(F)$) are replaced by their images in the homology group of F.] It is also asserted that a knowledge of the cohomology rings of M and F, and certain undefined generalizations of the characteristic isomorphisms, lead, in an unstated fashion, to a determination of the additive cohomology groups of $\bar{M}$.

N. E. Steenrod (Princeton, N. J.).

Eckmann, Beno. Coverings and Betti numbers. Bull. Amer. Math. Soc. **55**, 95–101 (1949).

Let P, $\bar{P}$ be finite polyhedra, P a regular covering of $\bar{P}$, and G the group of covering transformations. For each n, the group G has a natural representation R_n as a group of linear transformations of the nth homology group $H_n(P)$ with real coefficients. The subgroup of elements invariant under G is proved to be isomorphic to $H_n(\bar{P})$. This yields the result $g\bar{p}_n=\sum s_n(x)$ over $x\in G$, where g is the order of G, $\bar{p}_n$ is the nth Betti number of $\bar{P}$, and $s_n(x)$ is the character of R_n. In this way the Betti numbers of $\bar{P}$ are determined by the representations R_n of G. In case P is an orientable N-manifold, it is shown that $s_n(x)=\gamma s_{N-n}(x)$, where γ is the degree of x. This is applied to the case $\bar{P}$ is a nonorientable N-manifold and P is the 2-fold orientable covering to yield a number of relations between the Betti numbers p_n, $\bar{p}_n$ of P, $\bar{P}$. Most interesting is the duality relation $\bar{p}_n+\bar{p}_{N-n}=p_n$. Also, if $N=3$, $p_1=2\bar{p}_1-1$; if $N=2L$, $p_L=2\bar{p}_L$.

N. E. Steenrod (Princeton, N. J.).

Spanier, Edwin H. The Mayer homology theory. Bull. Amer. Math. Soc. **55**, 102–112 (1949).

W. Mayer defined homology groups [Ann. of Math. (2) **43**, 370–380, 594–605 (1942); these Rev. **3**, 318] based on a boundary operator ∂ such that $\partial^p=0$ (p a prime) instead of the usual $\partial^2=0$. He obtained a doubly indexed system of groups $H_{n,q}$ $(0<q<p)$ based on a coefficient group having only elements of order p. The author shows that most of the Mayer groups are zero and the remainder are isomorphic to ordinary homology groups. Thus, for any complex: (1) if $n\equiv q-1 \pmod p$, $H_{n,q}\approx H_r$ for $r=2(n-q+1)/p$, (2) if $n\equiv -1 \pmod p$, $H_{n,q}\approx H_r$ for $r+1=2(n+1)/p$, (3) in all other cases, $H_{n,q}=0$. These results show that the groups of Mayer do not provide any new topological invariants.

N. E. Steenrod (Princeton, N. J.).

Spanier, E. Borsuk's cohomotopy groups. Ann. of Math. (2) **50**, 203–245 (1949).

Section I deals with the definition and some properties of the Borsuk (or cohomotopy) groups $\pi^n(X, A)$ for a pair (X, A) consisting of a compact normal Hausdorff space X and a closed $A\subset X$. Let S_n be the unit n-sphere, and $p\in S_n$; inverting the procedure used to define the homotopy groups π_n, consider instead the set of all $f: X\to S_n$ satisfying $f(A)=p$. The relation of homotopy rel A divides all these mappings into homotopy classes, and this collection of homotopy classes is denoted by $\pi^n(X, A)$. It is shown that, if $\dim(X-A)<2n-1$, then $\pi^n(X, A)$ can be regarded as an (always) Abelian group and, in particular, $\pi^n(S_m, p)=\pi_m(S_n, p)$ for $m<2n-1$.

An $f: X\to Y$, $f(A)\subseteq B$ induces, under the conditions stated on the dimensions, a homomorphism $f^+: \pi^n(Y, B)\to\pi^n(X, A)$. Also, a homomorphism Δ, the "coboundary operator", can be constructed so that

$$\cdots\to\pi^n(X)\xrightarrow{i^+}\pi^n(A)\xrightarrow{\Delta}\pi^{n+1}(X, A)\xrightarrow{j^+}\pi^{n+1}(X)\to\cdots,$$

i^+, j^+ induced by identity maps, is an exact sequence. Using f^+, i^+, j^+, Δ, it is shown that the Borsuk groups of a pair (X, A), under the additional hypothesis that $\dim X<2n-1$ satisfy all the Eilenberg-Steenrod axioms for a cohomology theory, except their existence for every n. (This includes the excision axiom 6, which is the only one that the homotopy groups do not satisfy.)

If $\dim(X-A)<2n-1$, there is always a homomorphism of $\pi^n(X, A)$ into the Čech cohomology group $H^n(X, A; I)$ of X mod A over the integers. Under the additional hypotheses that X is a cell-complex, A a subcomplex, and X^m the m-dimensional skeleton of X, the author shows that $\pi^n(X^m\cup A, X^{m-1}\cup A)=C^m(X, A; \pi_m(S_n))$, where $m<2n-1$, and $C^m(X, A)$ is the group of cochains of X mod A. Finally, for any pair (X, A) satisfying $\dim X<2n-1$, the Borsuk groups can also be defined as follows: let $\{N_\nu, L_\nu\}$ be the spectrum of polytopes yielding $H^n(X, A)$; then

$$\pi^n(X, A)=\varinjlim\pi^n(N_\nu, L_\nu).$$

(This "continuity" property is not true for the homotopy groups, which are related instead to the singular homology groups.)

Section II gives an application. Clearly, if $\pi^n(X)$ can be computed, we have at hand a classification theorem for maps of the space X into S_n. The author confines himself to polytopes, and using Steenrod's squaring operation, constructs an exact sequence of groups that involves both the cohomotopy and cohomology groups. It is shown that the exactness of this sequence implies (in fact, is equivalent to) all the known classification theorems for maps of an m-polytope into an n-sphere [Hopf, for maps of $K^n\to S_n$, and Steenrod for maps of $K^{n+1}\to S_n$ with $n>2$]. Pontrjagin's classification of maps of $K^3\to S_2$ is not included, because the set $\pi^3(K^3)$ involved is not a group. It is further shown that, for $n>2$, $\pi^{n+1}(K^n)$ is, in fact, a group extension of an effectively calculable group by an effectively calculable group.

J. Dugundji (Los Angeles, Calif.).

Whitehead, J. H. C. On simply connected, 4-dimensional polyhedra. Comment. Math. Helv. **22**, 48–92 (1949).

It is shown that the homotopy type of a 4-dimensional simply connected complex P is determined by its interrelated "cohomology rings." By the latter are meant the cohomology rings based on integer coefficients, and on integers mod r for each r, the homomorphisms connecting these rings induced by the natural coefficient group homomorphisms, a "coboundary operator Δ" attached to each coefficient group homomorphism, and finally the Pontrjagin squaring operation which to each n-cocycle mod $2r$ assigns a $2n$-cocycle mod $4r$. The entire algebraic structure, denoted by $R(P)$, is called the ring of P. Necessary and sufficient conditions are given for an abstract "ring" to be the $R(P)$ of some simply connected 4-complex P. A notion of "proper" homomorphism of one such ring into another is defined, and it is shown that a homomorphism $R(P_1)\to R(P_2)$ is proper if and only if it is induced by a map $f: P_2\to P_1$. It is then shown for simply connected complexes of arbitrary dimensions that a map $f: P_2\to P_1$ is a homotopy equivalence if f induces isomorphisms of the cohomology groups based on integer coefficients. An incidental result is that the homotopy type of a simply connected 3-complex is determined by its cohomology groups based on integer coefficients.

A notion of reduced complex is defined. It consists of a cluster of 2-spheres and 3-spheres with a single common point, a collection of 3-cells each having for boundary a multiple of some 2-sphere, and a collection of 4-cells whose

boundaries lie on the union of the spheres. Any simply connected 4-complex is proved to be homotopically equivalent to a reduced complex. This permits restricting the discussion to reduced complexes. If P_1, P_2 are reduced, isomorphisms of their second and third cohomology groups (integers) are readily realized by a map of the 3-skeleton. The difficulties in extending over the 4-cells lie in the way their boundaries are mapped on the 2-spheres. The various ways they may be attached are differentiated by cup products and Pontrjagin squares of 2-cocycles.

The homotopy classification problem of maps of one simply connected 4-complex into another is not solved, even for the case where the second complex is a 2-sphere. Two maps may induce the same homomorphism of the ring without being homotopic. It is only in the special case that the homomorphism is an isomorphism that the solution is given. Then they are homotopic, and, in fact, are homotopy equivalences. *N. E. Steenrod* (Princeton, N. J.).

Whitehead, J. H. C. On the realizability of homotopy groups. Ann. of Math. (2) **50**, 261–263 (1949).

The author shows that the homotopy groups π_i, $i=1, 2, \cdots$ (with π_i, $i>1$, Abelian and π_1 operating on π_i, $i>1$) of a polyhedron can be prescribed at will. The polyhedra used have to be allowed to be infinite-dimensional; otherwise the theorem is not true. The proof proceeds by constructing the polyhedron with spheres and cells corresponding to generators and relations of suitable homotopy groups, and makes use of previous results of the author. If the π_i are countable, the polyhedron can be chosen locally finite.
H. Samelson (Ann Arbor, Mich.).

Postnikov, M. M. The classification of continuous mappings of a three-dimensional polyhedron into a simply connected polyhedron of arbitrary dimension. Doklady Akad. Nauk SSSR (N.S.) **64**, 461–462 (1949). (Russian)

The title of this note is in error: it is the three-dimensional polyhedron which is assumed simply connected. The paper gives a necessary and sufficient condition for the equivalence in the sense of homotopy of two mappings of a simply connected, connected, three-dimensional complex K into a polyhedron Y. The condition is in terms of certain cochains on K with coefficients in $\pi^2(Y)$ and $\pi^3(Y)$. A continuous map $f(K)$ of K into Y is called normal if every point on the one-simplexes of a given, but arbitrary, triangulation of K is mapped into the same point of Y. A pair of normal maps, f and g, constitute a normal pair if they coincide on the set of all two-simplexes of K. If f is normal, and s^3 is a two-cell of K, associate with f and s^3 the element $d(f, s^3)$ of $\pi^3(Y)$. This defines a characteristic cocycle $d^3(f)$ over K, coefficients in $\pi^3(Y)$. A normal f and g are equivalent to a normal pair if and only if $d^3(f)=d^3(g)$.

If f and g are a normal pair and s^3 is a three-cell of K there is associated with them an element $d(f, g: s^3)$ of $\pi^3(Y)$. This element is the coefficient of a cocycle defined on K, values in $\pi^3(Y)$, denoted by $d^3(f, g)$. Now, f and g are homotopic if and only if there exists on K a one-dimensional cocycle e^1, with coefficients in $\pi^3(Y)$, such that

$$d^3(f, g)=(e^1)\cup(d^2)+(e^1)*(e^1);$$

in this resolution of the cocycle d^3 the cocycle d^3 is the $d^3(f)=d^3(g)$, defined above. Each summand on the right side is a product of cochains. The symbol $\cup$ is used for a product in the sense of Alexander, based on an ordering of the vertices of K; in this product, the product of two elements of $\pi^2(Y)$ is the appropriate element of $\pi^3(Y)$ as defined by J. H. C. Whitehead [Ann. of Math. (2) **42**, 409–428 (1941); these Rev. **2**, 323]. The definition of the second summand is somewhat involved, and depends among other things on the decomposition of the group $\pi^2(Y)$ into its direct factors. The notation $(e^1)*(e^1)$ merely emphasises the dependence of this term upon the homology class only of the chosen cocycle e^1. *L. Zippin* (Brooklyn, N. Y.).

☆ Please see Errata, p. 870, V.11.9 Reviews.

Lokucievskiĭ, O. On open mappings of plane compacta. Doklady Akad. Nauk SSSR (N.S.) **64**, 625–628 (1949). (Russian)

Let T denote a triod (sum of three segments with a common vertex) and let ωT denote the topological product of T by an infinite point set homeomorphic to the following sequence of reals: 0, 1, 1/2, 1/3, $\cdots$, 1/n, $\cdots$. The set ωT is the sum of an infinite set of triods, and one of these is a sequential limit set for the system. The author proves the following results. If a plane compactum X can be mapped quasimonotonically on a triod T, then X has at most a finite number of components. Principal theorem: no plane compactum can be mapped quasimonotonically on a compactum which contains ωT. Corollary: if at least one component of a given compactum X is homeomorphic to a plane set, then the compactum X cannot be mapped quasimonotonically onto any continuum (e.g., the cube) which contains the topological image of ωT. *L. Zippin.*

Haupt, Otto. Zur Struktur der Kompakta von endlicher Ordnung. Math. Ann. **120**, 423–429 (1949).

It is known that if E is a compact set in Euclidean n-space E^n, $n\geqq 2$, which is of finite order relative to the set of all k-planes, $1\leqq k\leqq n-1$, then (I) E is at most k-dimensional, and (II) E is the union of two sets E' and O, where O is open in E and at most $(k-1)$-dimensional and where E' is the closure of the union of a sequence $\{Z_i\}$ of disjoint k-cells, each open in E'. The author makes a further study of these k-cells. He shows that if O is vacuous, so that $E'=E$, then the cells Z_i can be chosen to satisfy the following condition. For each i there is a coordinate system $(x_1^{(i)}, \cdots, x_n^{(i)})$ in E^n such that Z_i is the image of a fixed reference k-cell under a mapping $g^{(i)}=(g_1^{(i)}, \cdots, g_n^{(i)})$ which satisfies a Lipschitz condition. From this it follows that E' has a dense subset at each point at which E' has a tangent plane. This result fails if the set O is not vacuous. *E. G. Begle.*

Youngs, J. W. T. Remarks on cyclic additivity. Bull. Amer. Math. Soc. **55**, 427–432 (1949).

For a Peano space B denote by $E(B)$ the class of all true cyclic elements of B, and by $A(B)$ the class of A-sets in B. Define $P(B)$ as the class of all elements C such that C is a Peano subspace of B and $E(C)\subset E(B)$. Let $Q(B)$ be the class of all elements D with the property that D is a Peano subspace of B having a finite cyclic chain approximation $C_1, \cdots, C_q$ such that each C_k, $k=1, \cdots, q$, is an arc or an element of $E(B)$. It is seen that $A(B)$ and $Q(B)$ are subclasses of $P(B)$. Let X and Y be topological spaces. Then hk is a Peanian factorization of a mapping $f: X\to Y$ if and only if there are mappings $k: X\to B$, $h: B\to Y$ such that B is a Peano space and $f=hk$. The space B is called the middle space of the Peanian factorization hk of f. Let G be a commutative topological semi-group (with zero element) which, as a space, is Hausdorff. Denote by F the

class of all mappings $f: X \to Y$ each of which has at least one Peanian factorization, and suppose ω is a transformation from F into G. If $E \in E(B)$, then there exists a unique monotone retraction of B onto E, denoted by r_E. If $f \in F$ and hk is a Peanian factorization of f with middle space B while $E \in E(B)$, define $f_E = hr_E k$. The mapping f_H for $H \in P(B)$ is also defined, the definition being similar but more complicated.

The transformation ω is said to be cyclicly additive provided $\omega(f) = \sum \omega(f_E)$, $E \in E(B)$, for each f in F and each Peanian factorization of f. The equality means that for each neighborhood U of $\omega(f)$ there is a finite subclass $I(U)$ of $E(B)$ such that if I is any finite subclass of $E(B)$ containing $I(U)$ then $U \supset \sum \omega(f_C)$, $C \in I$, it being understood that addition over an empty class yields 0. It is shown that if ω is cyclicly additive then the following conditions hold. (I) If $f \in F$ and has a Peanian factorization in which the middle space is a dendrite, then $\omega(f) = 0$. (I_a) If $f \in F$ and has a Peanian factorization in which the middle space is an arc, then $\omega(f) = 0$. (II) If $f \in F$ has a Peanian factorization hk with middle space B while $A_i \in A(B)$, $i = 1, 2$, $A_1 \cap A_2$ is a point, $A_1 \cup A_2 = B$, then $\omega(f_{A_1}) + \omega(f_{A_2}) = \omega(f)$. ($II_a$) The same as (II) except that the hypothesis is strengthened by the condition $B \in Q(B)$. (III) If $f \in F$ and has a Peanian factorization hk with middle space B, then for each neighborhood U of $\omega(f)$ there is a Q in $Q(B)$ such that $Q \subset P \in P(B)$ implies $\omega(f_P) \in U$. (III_a) If $f \in F$ and has a Peanian factorization hk with middle space B, then for each neighborhood U of $\omega(f)$ there is a $Q \in Q(B)$ such that $Q \subset R \in Q(B)$ implies $\omega(f_R) \in U$. It is also shown that the combination of the conditions (I_a), (II_a), (III_a) is sufficient to make ω cyclicly additive. The paper closes with an application to the Čech cohomology groups. *D. W. Hall* (College Park, Md.).

Young, Gail S., Jr. On 1-regular convergence of sequences of 2-manifolds. Amer. J. Math. 71, 339–348 (1949).

A well-known theorem of Whyburn states that if a sequence of 2-spheres converges 1-regularly to a nondegenerate set then the limit set is again a 2-sphere. The reviewer extended this theorem to the case of compact orientable 2-manifolds and also showed that the limit set is homeomorphic to almost all elements of the sequence. In this paper Young disposes of the case of sequences of general 2-manifolds. Here it is necessary to have two kinds of regular convergence: (I) $\{A_n\}$ converges W-k-regularly to A if (a) $\lim A_n = A$ and (b) $\{A_n\}$ converges k-regularly at each point of A; (II) $\{A_n\}$ converges S-k-regularly to A if (a) $\rho_H(A_n, A) \to 0$, where ρ_H is Hausdorff distance, and (b) $\{A_n\}$ converges k-regularly uniformly over A. If $\{A_n\}$ is a sequence of 2-manifolds, the author shows that A is a 2-manifold if $\{A_n\}$ converges W-1-regularly to A and $\{B_n\}$ converges W-0-regularly to B or else has an empty upper limit, where B_n is the boundary of A_n. However, there are examples to show that in this case A need not be homeomorphic to any A_n. But if $\{A_n\}$ converges S-1-regularly to A and $\{B_n\}$ converges S-0-regularly to B, then A is homeomorphic to all but a finite number of the A_n.

E. G. Begle (New Haven, Conn.).

Young, Gail S., Jr. On continuous curves irreducible about compact sets. Bull. Amer. Math. Soc. 55, 439–441 (1949).

If S is a convex metric space and if K is a compact set whose components are locally connected and form a null family then S contains a Peano curve irreducible about K. This theorem, together with results announced by R. H. Bing and E. E. Moise, solve a problem proposed by L. Zippin. *A. D. Wallace* (New Orleans, La.).

GEOMETRY

Bachmann, Friedrich. Geometrien mit euklidischer Metrik, in denen es zu jeder Geraden durch einen nicht auf ihr liegenden Punkt mehrere Nichtschneidende gibt. III. Math. Nachr. 1, 258–276 (1948).

[The first two parts seem not to have appeared yet.] A metric Euclidean geometry is a geometry which satisfies (i) the axioms of absolute geometry as given by A. Schmidt [Math. Ann. 118, 609–625 (1943); these Rev. 6, 13] and (ii) the condition that every quadrilateral containing three right angles is a rectangle. A Euclidean geometry satisfies (i) and (ii) and the parallel axioms. A metric Euclidean geometry which is a subgeometry of a Euclidean geometry is said to belong to the Euclidean geometry provided all lines passing through a point P of the subgeometry are lines of the subgeometry. In parts I and II, the author considered the problem of characterising algebraically the metric Euclidean geometries which belong to a Euclidean geometry, assuming first that every right angle is bisectable and then that every angle is bisectable. In the present paper the problem is considered without supplementary assumptions.

An orthogonality constant k is defined as follows. If a Euclidean geometry G is represented as a coordinate geometry over a field K and if k is an element of K such that $-k$ is not a square in K, then we may agree that the orthogonality of the lines $u'x+v'y+w'=0$ and $u''x+v''y+w''=0$ is expressed by $u'u''+k^{-1}v'v''=0$. With this convention, G is referred to as a Euclidean geometry over the field K with orthogonality constant k. For such a geometry G, if the rotation δ is the product of reflections in the lines $y=0$ and $y=cx$ ($c \neq 0$ in K), then the transformation equations representing the half rotation $h(\delta)$ are

$$x^* = \frac{1}{1+kc^2}x - \frac{kc}{1+kc^2}y,$$

$$y^* = \frac{c}{1+kc^2}x + \frac{1}{1+kc^2}y.$$

By insisting on closure under this and similar transformations, the following criterion for abscissa sets is established. A subset A of K is the abscissa set of a metric Euclidean subgeometry which belongs to G, if and only if (1) A contains at least one element other than 0; (2) a and a' in A implies $a-a'$ in A; (3) a in A implies

$$\frac{1}{1+kc^2}a, \quad k\frac{c}{1+kc^2}\,\frac{c'}{1+kc'^2}a$$

in A (c and c' in K).

The following rings are defined: $A_k(K)$ consists of all finite sums of products

$$\pm k^m \prod^{2m} \frac{C_\mu}{1+kC_\mu^2} \prod^{n} \frac{1}{1+kC_\nu^2},$$

C_μ, C_ν in K; $m, n = 0, 1, 2, \cdots$; $\mu = 1, 2, \cdots, 2m$; $\nu = 1, 2, \cdots, n$; $B_k(K)$ consists of all finite sums of products

$$\pm k^m \prod^{2m+1} \frac{C_\mu}{1+kC_\mu^2} \prod^{n} \frac{1}{1+kC_\nu^2},$$

C_m, C_r in K; $m, n = 0, 1, 2, \cdots$; $\mu = 1, 2, \cdots, 2m+1$; $\nu = 1, 2, \cdots, n$. It is shown that $A_k(K)$ and $B_k(K)$ are respectively the abscissa and ordinate sets of the smallest metric Euclidean subgeometry belonging to G and containing the points $O(0,0)$ and $E(1,0)$. The ring $A_k(K)$ contains, as a subring, the ring of elements $C_k(K)$ consisting of all finite sums of products $\pm\prod^n 1/(1+kC_\nu^2)$, C_ν in K, $n = 0, 1, \cdots$; $\nu = 1, 2, \cdots, n$. The question of the equality $A_k(K) = B_k(K) = C_k(K)$ is discussed.

When the field K is replaced by the field of rational numbers P, the constant k is further restricted to be an integer which is not a square and is different from 0 and -1. Then it is shown that $C_k(P)$ consists of rational numbers which, in reduced form, contain in their denominators only prime factors for which $-k$ is a quadratic residue. Further, $C_k(P)$ is the abscissa and ordinate set of the smallest metric Euclidean subgeometry which belongs to the Euclidean geometry over the field P with orthogonality constant k, and contains O and E.

The Euclidean geometry with orthogonality constant 1, of which the coordinate field is the quadratic number field $P(\sqrt{k})$, where k is an integer, not a square, unequal to $0, 1, -1$, is considered. In this geometry, it is shown that points $(a, b\sqrt{k})$, a, b in P, and their connecting lines, form a subgeometry which is isomorphic to Euclidean geometry over the field P with orthogonality constant k. [If $k>0$, this subgeometry is also a subgeometry of ordinary real Euclidean geometry.] The ring $C_1(P(\sqrt{k}))$, produced by elements $1/(1+c^2)$ with c in $P(\sqrt{k})$, is seen to be (i) the coordinate set of the smallest metric Euclidean subgeometry containing O and E and belonging to Euclidean geometry over the field $P(\sqrt{k})$ with orthogonality constant 1 and (ii) the ring consisting of $C_1(P)$ and $C_k(P)$ expanded through the adjunction of $\sqrt{k}$.

The following question is posed. If p and q are orthogonal vectors in a Euclidean plane, what vectors are obtained by proceeding from p and q by finite sequences of the two processes: (1) from the vectors a and b form the difference vector $a-b$ and (2) from the vectors a, b with $b \neq 0$, form the projection a_b? The answer is provided in the following results. (a) If $p = \overrightarrow{OX}$, $q = \overrightarrow{OY}$ are two orthogonal vectors in a Euclidean geometry, consider the end-points of the vectors produced from p and q by means of processes (1) and (2). These end-points and their connecting lines constitute the smallest metric Euclidean subgeometry containing O, X, Y. (b) In real Euclidean geometry, a vector a can be obtained from the vectors $\overrightarrow{OE}$ and $\overrightarrow{OY}_1$, with $Y_1 = (0, \sqrt{k})$, by the processes (1) and (2), if and only if the x-component of a belongs to $C_k(P)$ and the y-component to the module $\sqrt{k}C_k(P)$. (c) In real Euclidean geometry, a vector a can be obtained from the vectors $\overrightarrow{OE}$ and $\overrightarrow{OY}_2$ with $Y_2 = (0, 1+\sqrt{k})$ by the processes (1) and (2), if and only if the x- and y-components of a belong to $C_1(P(\sqrt{k}))$. *L. Dulmage.*

Pepper, Paul M. A new method in imbedding theorems. Rep. Math. Colloquium (2) **8**, 39–48 (1948).

The imbedding theorems dealt with in this paper concern congruent imbedding of an arbitrary semi-metric space S in Euclidean and spherical n-spaces. It is well known (1) that S is imbeddable in E_n or S_n provided each $(n+3)$-tuple of S is, and (2) in order to establish the result it is not necessary to assume that all $(n+3)$-tuples have that property (i.e., a certain number of $(n+3)$-tuples might be left free). The author re-establishes previous results concerning the number of such $(n+3)$-tuples [Pepper and Topel, same Rep. (2) 4, 31–55 (1943); these Rev. 5, 46] by a method based on a theorem in elementary combinatorial analysis which he establishes for that purpose. *L. M. Blumenthal.*

Steiner, Jacob. Geometrical constructions with a ruler, given a fixed circle with its center. Translated from the first German edition (1833) by Marion Elizabeth Stark. Edited with an introduction and notes by Raymond Clare Archibald. Scripta Math. **14**, 187–264 (1 plate) (1948).

The introduction contains bibliographical and biographical material.

v. Sz. Nagy, Gyula (Julius). Ein Beweis des Satzes von H. J. S. Smith und H. Kortum. Elemente der Math. **3**, 95–97 (1948).

Simplified proof of the following theorem: the real roots of any cubic or biquadratic equation with real coefficients can be constructed by means of straight edge, compasses and a given fixed conic different from a circle. [For an earlier elementary proof see Vahlen, Arch. Math. Phys. (3) **3**, 112–120 (1902).] It is noted that the words "real" in the statement of the theorem may be omitted.

F. A. Behrend (Melbourne).

Nestorovič, N. M. On the quadrature of the circle and the circularization of the square in Lobachevskian space. Doklady Akad. Nauk SSSR (N.S.) **63**, 613–614 (1948). (Russian)

(1) Circularization of the square. The following equation determining the radius of the equivalent circle may be derived: $\sinh \frac{1}{2}R/k = (\frac{1}{2} - \frac{1}{4}\omega)^{\frac{1}{2}}$. In this formula $\pi\omega$ represents the sum of the angles of the square. The construction is only possible when ω is a number derived from rationals by a finite number of rational operations and taking the square root. (2) Quadrature of the circle. The acute angle α of the square may be expressed as follows: $\alpha = \pi - \frac{1}{2}\pi \cosh R/k$. The construction is only possible when $\cosh R/k$ represents a rational fraction m/n, where the denominator n is an integer built up from Gaussian primes and factors 2^i. Thus the quadrature of the circle is equivalent to the problem of "Kreisteilung." *H. A. Lauwerier* (Amsterdam).

Tortorici, M. Rapporto tra proiezione ortogonale di un segmento e segmento obiettivo in geometria piana iperbolica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 116–119 (1948).

Let $AA'B'B$ be a trirectangle with right angles at A, A', B', and let the parallel to AB from A' meet BB' in C'. Then the segment $l' = A'B'$ is the orthogonal projection of the segment $l = AB$. The author defines the angle β between the ultraparallel lines AB and $A'B'$ to be the complement of $\Pi(AA')$; thus β is the angle at A' in the right triangle $A'B'C'$ whose hypotenuse $A'C'$ is equal to l by Bolyai's construction for parallels [cf. Sevdić, Hrvatsko Prirodoslovno Društvo Glasnik Mat.-Fiz. Astr. Ser. II. **2**, 11–17 (1947); these Rev. **10**, 139]. Hence $(\tanh l')/(\tanh l) = \cos\beta$, $\lim_{l\to 0}(l'/l) = \cos\beta$, and

$$l(d/dl)(l'/l) = (\sinh 2l'/\sinh 2l) - (l'/l) < 0.$$

The author obtains these results more laboriously.

H. S. M. Coxeter (Toronto, Ont.).

Haupt, Otto. Zur geometrischen Kennzeichnung der Scheitel ebener Kurven. Arch. Math. **1**, 102–105 (1948).

The circular order [c.o.] of an arc A is the maximum number of points in which any circle meets A. The c.o. of a

point on A is defined to be the c.o. of all sufficiently small neighborhoods on A of this point. An S-vertex is a point of a c.o. greater than three. At an E-vertex, the radius of curvature has a finite extremum not equal to zero. The first of the three theorems of this note gives various properties of arcs of c.o. 3. In particular, if A has no E-vertices and satisfies certain smoothness assumptions, then A has the c.o. 3. Under similar assumptions, every E-vertex on A is an S-vertex, and every isolated S-vertex on A is an E-vertex. In the last theorem, the isolated S-vertices on such an arc A are characterized in different ways. They are the points of c.o. 4. *P. Scherk* (Saskatoon, Sask.).

Haupt, Otto. Über die Krümmung ebener Bogen endlicher Ordnung. S.-B. Phys.-Med. Soz. Erlangen **71** (1939), 219–227 (1940).

The linear order of an arc is the maximum number of its points of intersection with a straight line. Let B denote an arc with a continuous tangent. The curvature of B is defined by means of the intersection of neighboring normals. If the order of B is not greater than three, then the points at which B has no finite curvature form a set of measure zero. This follows at once from the fact that B is then a finite sum of convex arcs. The author constructs an arc B of order four which contains a perfect nowhere dense subset S with the following properties: the measure of S differs arbitrarily little from the length of B, and at none of the points of S does B possess a curvature. *P. Scherk.*

Nicolesco, Miron. Sur un lemme de M. Pompeiu. Bull. Math. Phys. Êc. Polytech. Bucarest **10** (1938–39), 22–26 (1940).

Consider vectors in Euclidean n-space. Two such vectors are called acute or obtuse according as their scalar product is positive or negative. It is shown that if some n of $n+1$ given pairwise obtuse vectors form a basis, then each coordinate with respect to this basis of the remaining vector is nonpositive. [It may, however, easily be demonstrated that each such coordinate is actually negative.] As a consequence, any nonnull vector is acute with some one of $n+1$ given pairwise obtuse vectors. [From this it follows that any n of $n+1$ given pairwise obtuse vectors form a basis.] *W. Gustin* (Bloomington, Ind.).

Arghiriade, E. La transformation par centres radicaux. Bull. Sci. Tech. Polytech. Timişoara **13**, 38–53 (1948).

Étant donnés 3 cercles de base C_1, C_2, C_3, on peut associer à tout point M du plan 3 couples de cercles γ_1', γ_1''; γ_2', γ_2''; γ_3', γ_3'', de rayons ρ_i', ρ_i'' $(i=1, 2, 3)$, respectivement concentriques aux cercles de base, et tels que les triples de cercles $(\gamma_1', C_2, \gamma_3'')$, $(\gamma_2', C_3, \gamma_1'')$, $(\gamma_3', C_1, \gamma_2'')$ aient chacun M pour centre radical. A chacun de ces 3 couples, on associe un nouveau cercle concentrique (respectivement Γ_1, Γ_2, Γ_3), de rayons ρ_i définis par une même combinaison linéaire $\rho_i^2=\varepsilon\rho_i'^2+\varphi\rho_i''^2+\tilde{\omega}$. Le triple $(\Gamma_1, \Gamma_2, \Gamma_3)$ a pour centre radical un point M_1. L'auteur étudie en détail la transformation pseudo-symétrique $(\varphi=-\varepsilon;\ \tilde{\omega}=0)$, qui établit une affinité entre M et M_1, et la transformation symétrique $(\varphi=\varepsilon)$, qui établit une homothétie entre M et M_1. Ces transformations, et les correspondances entre axes radicaux, font intervenir de nombreux cercles et coniques remarquables liés au triangle des centres. *P. Belgodère.*

Kadeřávek, František. Contribution to the study of normals on quadric surfaces. Časopis Pěst. Mat. Fys. **73**, D46–D49 (1949). (Czech)

Ghosh, N. N. Rigid rotation in hyperspace. Bull. Calcutta Math. Soc. **40**, 116–122 (1948).

A vector $(x_1, x_2, \cdots, x_n)$ determines a matrix X of order $n+1$ having $0, x_1, x_2, \cdots, x_n$ in the first column, and $0, -x_1, \cdots, -x_n$ in the first row, and zero everywhere else. This device enables the author to express a proper orthogonal transformation (in Euclidean n-space) in the form $X'=e^{\Omega}X+Xe^{-\Omega}$, where Ω is a skew matrix of order $n+1$ having zero throughout the first row and column. In the case when $\Omega=AB-BA$, where A and B are two matrices of the same type as X, this turns out to be a rotation in the plane of the vectors represented by A and B. The general proper orthogonal transformation, involving simultaneous rotations in r completely orthogonal planes $(r\leqq\frac{1}{2}n)$, is obtained by setting Ω equal to the sum of r matrices like $AB-BA$, one for each plane. [See the author's earlier paper, same Bull. **35**, 115–125 (1943); these Rev. **6**, 103.] This treatment is applied to the problem of compounding two rotations whose planes have a line of intersection. *H. S. M. Coxeter* (Toronto, Ont.).

Zacharias, Max. Eine neue ebene Konfiguration $(12_4, 16_3)$. Math. Nachr. **1**, 332–336 (1948).

Continuing his previous paper [Deutsche Math. **6**, 147–170 (1941); these Rev. **8**, 219], the author obtains another such configuration of 12 points on 16 lines by projecting the points and lines of the special complete hexahedron formed by three coaxal planes and three planes of general position. By disregarding 4 of the 16 lines he derives a self-dual 12_3; but this is less interesting than it might seem, as some of the 12 points (and likewise some of the 12 lines) differ essentially from others. *H. S. M. Coxeter* (Toronto, Ont.).

Todd, J. A. Poncelet's poristic polygons. Math. Gaz. **32**, 274–280 (1948).

What is the condition for two conics S and S' to be so situated that an n-gon can be inscribed in S' and circumscribed about S? Cayley [Collected Mathematical Papers, v. 2, pp. 87–90 = Philos. Mag. (4) **6**, 99–102 (1853)], using elliptic functions, expressed it as the vanishing of the determinant of order $[(n-1)/2]$ whose typical element is c_{i+j} or c_{i+j+1} according as n is odd or even, c_r being the coefficient of λ^r in the expansion of the square root of the discriminant of the conic $S+\lambda S'$. The author obtains the same result in an elementary manner. *H. S. M. Coxeter.*

Signorini, Antonio. Un théorème anallagmatique en cinématique des surfaces. C. R. Acad. Sci. Paris **228**, 160–162 (1949).

L'auteur étudie les ondes, issues à l'instant $t=0$ d'un épicentre E, se propageant dans un milieu à symétrie sphérique de centre O (isotrope autour de chacun de ses points, mais hétérogène lorsque la distance r à O varie), avec une vitesse de propagation $u=u_0(r/r_0)^{1-\mu}$. Par la transformation conforme $r'/r_0=(r/r_0)^{\mu}$, $\theta'=\mu\theta$ dans l'un des plans méridiens issus de OE, le problème est ramené au cas banal $\mu'=1$ des milieux homogènes et isotropes, avec des ondes circulaires de centre E. Il en résulte que, pour $0<t<r_0/u_0|\mu|$, la surface d'onde σ_t est invariante par l'une des inversions de pôle O et de puissance positive. Pour $1/\mu$ entier, il subsiste une telle inversion, de puissance positive ou négative selon que $1/\mu$ est pair ou impair, pour $t>r_0/u_0|\mu|$. Le cas limite $\mu=0$ est traité directement. *P. Belgodère* (Paris).

Glagolev, A. A. New applications of generalized throws. Doklady Akad. Nauk SSSR (N.S.) **62**, 285–287 (1948). (Russian)

The synthetic theory of the tetrahedral complex may be simplified considerably when the notion of generalized throws is introduced. In addition, this permits the theory to be extended at once to n-dimensional space. The author gives several generalizations of well-known theorems for the three-dimensional space. Short proofs are given.

H. A. Lauwerier (Amsterdam).

Labra, M. Analytic study of a homothety in the triangle. Revista Ci., Lima **50**, 39–68 (1948). (Spanish)

Thébault, Victor. Un chapitre de la géométrie récente du tétraèdre: symédianes et second point de Lemoine. Bull. Soc. Math. France **76**, 95–107 (1948).

Gambier, B. Sur les tétraèdres dont certaines bihauteurs se rencontrent. Bull. Soc. Math. France **76**, 79–94 (1948).

The author discusses the cases when (1) the two bialtitudes of a skew quadrilateral are coplanar, (2) one bialtitude of a tetrahedron is coplanar with each of the other two, (3) the three bialtitudes are coplanar. The method of presentation makes little or no appeal to the reader's knowledge of ruled quadric surfaces, or the theory of the tetrahedron. As indicated in the bibliography, the author's results have been obtained before by other, much less elementary, means. *N. A. Court* (Norman, Okla.).

Hadwiger, H. Bemerkung zur elementaren Inhaltslehre des Raumes. Elemente der Math. **4**, 3–7 (1949).

Let $V(P)$ be a real-valued function defined on all solid finite polyhedra in E^3 and satisfying the conditions: (1) $V(P) \geqq 0$, (2) $V(P+Q) = V(P) + V(Q)$ for P and Q without common interior points, (3) $V(P) = V(Q)$ for congruent P and Q, (4) $V(E) = 1$ for the unit cube E. Then $V(T) = Fh/3$ for a tetrahedron with base F and altitude h. This justifies the theory of elementary volume as published by S. O. Schatunovsky [Math. Ann. **57**, 496–508 (1903)] by supplying a uniqueness proof. *H. Busemann.*

van Dantzig, D. Plane axonometry of incidence constructions in the four-dimensional affine space. Nieuw Tijdschr. Wiskunde **35**, 141–150 (1947). (Dutch)

Convex Domains, Extremal Problems

Bol, Gerrit. Eilinien und sextaktische Punkte. Arch. Math. **1**, 94–101 (1948).

This is a projective proof of the theorem that a closed convex curve has at least six sextactic points. Its method is the same as that of the affine proof in Blaschke's "Vorlesungen über Differentialgeometrie" [v. 2, Springer, Berlin, 1923, pp. 43–44]. It may be of interest that in this projective version the curve is referred to an arbitrary parameter. *P. Scherk* (Saskatoon, Sask.).

Strubecker, K. Differentialgeometrie des isotropen Raumes. V. Zur Theorie der Eilinien. Math. Z. **51**, 525–573 (1949).

The theory of isotropic space developed by the author in the previous papers of this series [Akad. Wiss. Wien, S.-B. IIa. **150**, 1–53 (1941); Math. Z. **47**, 743–777 (1942); **48**, 369–427 (1942); **50**, 1–92 (1944); these Rev. **8**, 350; **7**, 530; **5**, 109; **8**, 96] is here applied to the study of ovals and pairs of ovals in the plane. A paratactic mapping of surface elements of isotropic space is introduced, transforming a surface strip of isotropic space into pairs of plane curves. Theorems regarding pairs of plane curves are thereby transformed into theorems on surface strips in isotropic space, and are then proved directly. Among the numerous theorems on ovals obtained by this method is, for example, the classical four-vertex theorem of relative differential geometry. Corresponding to a plane oval the author introduces a noteworthy surface of isotropic relative curvature -1, having the oval as an edge of regression, and the implications of the differential geometry of this surface for the oval are developed. Other topics studied by similar space methods include pairs of ovals having equal length or equal area, curves of constant width, Zindler curves, and characteristic properties of circles. *S. B. Jackson.*

Pogorelov, A. V. A general theorem of uniqueness for infinite convex surfaces. Doklady Akad. Nauk SSSR (N.S.) **65**, 131–133 (1949). (Russian)

The author's main theorem states in substance that, under certain circumstances, if two such surfaces agree asymptotically in a certain sense, either they coincide altogether, or their neighbourhoods of some pair P of parallel planes of support satisfy a relation which he terms "permitting strong inner contact." Unfortunately the author has suppressed part of his definitions and is only partly intelligible. If surfaces with corners are admitted his main theorem is false. In any case his proof is faulty for otherwise it would establish the corresponding result for every pair P. Counterexamples are obtainable by taking plane convex curves with a common pair of nonparallel asymptotes, each containing either a suitable corner or several circular arcs with different radii, suitably fitted together. *L. C. Young.*

Dinghas, Alexander. Über eine neue isoperimetrische Ungleichung für konvexe Polyeder. Math. Ann. **120**, 533–538 (1949).

Let P be a convex polyhedron in ordinary space and P^* the corresponding cap polyhedron, i.e., the polyhedron whose faces are tangent to the unit sphere S and parallel to the faces of P. By means of the so-called Wirtinger-Blaschke lemma the author proves the inequality $M^2 - OO^* \geqq 0$, where O and O^* denote the surface areas of P and P^*, respectively, and $M = \sum l_i \tan(\psi_i/2)$, where l_i is the length of an edge of P and ψ_i the angle of the outer normals of the faces joining at this edge. The author seems not to be aware that the above inequality is an immediate consequence of the Minkowski inequality $V(P, P^*, S)^2 \geqq V(P, P, S)V(P^*, P^*, S)$, for $O = 3V(P, P, S)$, $O^* = 3V(P^*, P^*, S)$ and, as is proved in the paper, $M \geqq 3V(P, P^*, S)$. *W. Fenchel* (Copenhagen).

Dinghas, A. Zur isoperimetrischen Ungleichung in Räumen konstanter Krümmung. Math. Z. **51**, 265–277 (1948).

Let V and O be the volume and surface of a solid $\mathfrak{K}$ in a non-Euclidean n-space R_n of curvature K, and let V_0 be the volume of the sphere in R_n of surface O. Thus (1) $V \leqq V_0$. The volume V_0 can be written as a definite integral V_0^* involving O, K, and the surface E_n of the unit sphere in Euclidean R_n. The author now constructs a quadratic polynomial $\varphi_K(x)$. Its coefficients depend on E_{n+1} and K, and it is monotonically increasing in the range of V and V_0.

Hence, (1) implies $\varphi_K(V) \leqq \varphi_K(V_0^*)$. The right hand term becomes a polynomial in $(O/E_n)^{1/(n-1)}$ if $K>0$ and $n\equiv 0 \pmod 2$ and if $K<0$ and $n\equiv 2 \pmod 4$. If $K>0$ or $K<0$ and $n\equiv 1 \pmod 2$, $\varphi(V_0^*)$ turns out to be an elementary transcendental expression.

In the last part of this paper, improvements of (1) are derived from the so-called linear isoperimetric inequality [cf., e.g., Math. Ann. 118, 636–686 (1943); these Rev. 6, 101]. One of them generalizes a refinement due to Bonnesen of the isepiphane inequality [Les Problèmes des Isopérimètres et des Isepiphanes, Gauthier-Villars, Paris, 1929, p. 135]. *P. Scherk* (Saskatoon, Sask.).

Dinghas, Alexander. Bemerkung zu einer Verschärfung der isoperimetrischen Ungleichung durch H. Hadwiger. Math. Nachr. 1, 284–286 (1948).

In a previous paper [Math. Z. 47, 669–675 (1942); these Rev. 7, 528] the author proved that in the isoperimetric inequality for a convex body K with corners in the n-dimensional space the volume of the unit sphere may be replaced by the volume of the cap body associated with K. In the present paper a still sharper inequality involving the radius of the inscribed sphere of K is proved by considering the inner parallel bodies of K. It generalizes and sharpens an inequality proved by Hadwiger [Elemente der Math. 3, 25–38 (1948); these Rev. 9, 526]. *W. Fenchel.*

Algebraic Geometry

Babbage, D. W. Twelve associated points in [5]. J. London Math. Soc. 23, 58–64 (1948).

The configuration of $2n+2$ associated points in S_n such that no $n+1$ lie in an S_{n-1} while any quadric primal through $2n+1$ of them contains the last has been treated by F. Bath, chiefly the particular case $n=4$. The author shows that a necessary and sufficient condition for $2n+2$ points of S_n to form such an associated set is that they lie at the vertices of two simplexes which are self-conjugate with respect to a unique nonsingular quadric primal, and that the system of such associated sets of points has $\frac{1}{2}n(3n+5)$ degrees of freedom. He treats the case $n=5$ in detail, finding that the normal elliptic sextic curve through any nine of the twelve associated points of S_5 has a trisecant plane determined by the remaining three points and that these three remaining points and the three intersections of the trisecant plane with the sextic lie on a conic. *T. R. Hollcroft* (Aurora, N. Y.).

Bílek, Jan. On a cubic involution in space, its degeneration and its application to the study of cubic surfaces. Časopis Pěst. Mat. Fys. 73, D37–D42 (1949). (Czech)

Burniat, Pol. Surfaces canoniques quadruples. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 34, 877–891 (1948).

For part I cf. the same vol., 239–251 (1948); these Rev. 10, 141.

Fano, Gino. Superficie del 4° ordine contenenti una rete di curve di genere 2. Pont. Acad. Sci. Comment. 7, 185–205 (1943).

The study of the quartic surface F^4 containing a sextic of genus two and, consequently, containing two nets of such sextics, mutually residual with respect to the intersections of F^4 with cubic surfaces was begun by the author and continued by Severi. Severi showed that an infinite discontinuous group of birational transformations is associated with F^4. In the present paper, a study is made of the quartic surface F^4 with a net of curves of genus two and of any order. If the order is $2n$, $n\geqq 3$, the birational transformations associated with F^4 form an infinite discontinuous group and on F^4 lie two nets of such curves mutually residual with respect to the intersections of F^4 with surfaces of order n. If the order is $2n+1$, $n\geqq 2$, and if (1) the associated Fermat-Pell equation has integral solutions, this net will be the only one on F^4 and the involution to which it belongs will be the only birational transformation to which F^4 is subject; on the other hand, if (2) the Fermat-Pell equation has only non-integral solutions, F^4 contains an infinity of nets of curves of order $2n+1$ and genus two and admits being transformed by an infinite, discontinuous group of birational transformations. *T. R. Hollcroft* (Aurora, N. Y.).

Galafassi, Vittorio Emanuele. Una questione di intersezione residua nell'iperspazio. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 10(79), 209–222 (1946).

The formula for the equivalence of a variety V_h common to a system of r hypersurfaces of orders N_i in S_r was derived by F. Severi [Mem. Accad. Sci. Torino (2) 52, 61–118 (1903), p. 105]. This formula is stated with the restriction $2h<r$ on the dimension of V_h. In the present paper, the author shows that the restriction $2h<r$ is not necessary: (1) when $N_i=N$, that is, the orders of the r hypersurfaces are all equal, and either V_h is or is not a complete intersection of $r-h$ hypersurfaces of order $n\leqq N$; (2) when the N_i are unequal, "presumably" in every case in which V_h is a complete intersection and in certain cases when V_h is not a complete intersection. *T. R. Hollcroft* (Aurora, N. Y.).

Galafassi, Vittorio Emanuele. Criteri di equivalenza nel gruppo jacobiano di un fascio di ipersuperficie. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 10(79), 345–361 (1946).

The Jacobian group J of a pencil of nonsingular hypersurfaces of order n in S_r contains $(r+1)(n-1)^r$ points. The author obtains the number of points of J absorbed by a singular point of the pencil or one of its surfaces. Two cases are considered, given the pencil $F\equiv\varphi+\lambda\psi=0$. First, P is a simple s-fold on φ but not on ψ, that is, P is not on the base variety; P absorbs $(s-1)^r$ points of J. In the second case, if Q is a simple s-fold base point of F, the number of points of J absorbed is $(rs+r+s-1)(s-1)^{r-1}$ if the tangent cone at Q is variable and $(rs-r+s+1)s^{r-1}$ if this tangent cone is fixed. If Q is also a simple t-fold point of φ $(t>s)$, and the tangent cone at Q is fixed, Q absorbs $(t+s-1)(t-1)^{r-1}+s(s-1)(r-1)(t-1)^{r-2}$ points of J. *T. R. Hollcroft* (Aurora, N. Y.).

Godeaux, Lucien. Sur certaines courbes fondamentales des transformations birationnelles de l'espace. Revista Ci., Lima 50, 31–37 (1948).

Jones, R. R. Some properties of a certain double surface in space of four dimensions. Proc. London Math. Soc. (2) 50, 380–389 (1948).

The normal rational curves of order four in four-space which pass through six points A_i fixed in general position and which in addition meet a given plane p generate a hypersurface V. Of these curves, those that meet p twice generate a surface S which is double for V. The surface S is studied in some detail. *A. Seidenberg.*

Piazzolla Beloch, Margherita. Sul numero delle secanti multiple e della classificazione delle curve algebriche sghembe. Ann. Univ. Ferrara **3**, 133–140 (1940).

The author derives a general formula for the number of multiple secants of multiplicities $4, 5, 6, \cdots, n-1$, of an algebraic space curve of given order n with no multiple points, and classifies space curves with respect to their multiple secants. She establishes the existence of all curves obtained by this method of classification and shows that for $n=7$ this method yields all existing types. A list of references to other methods of classification is given.

T. R. Hollcroft (Aurora, N. Y.).

Rollero, Aldo. Sul contatto del primo e del secondo ordine di due superficie algebriche nei punti di una loro retta comune. Euclides, Madrid **8**, 441–447 (1948).

This note deals with the points of contact of two algebraic surfaces of order n and m on a common line r on which the two surfaces have respectively p and q double points (or equivalent singularities). There are in general $n+m-(p+q+2)$ points of contact of the first order on r; if the two surfaces are tangent at all points of r then $m-n=q-p$, and that happens if there are $m+n-(p+q+1)$ points of contact. In this case there are $3n-2p+q-4$ points on r in which the two surfaces have a second order contact.

E. Bompiani (Rome).

Tanturri, Giuseppe. Una proprietà caratteristica delle C^r di S_n $(r\leqq n+1)$. Atti Sem. Mat. Univ. Modena **2**, 170–175 (1948).

Segre, B. Intorno agli S_k che appartengono alle forme generali di dato ordine. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 341–346 (1948).

It was already shown [part I, same vol., 261–265 (1948); these Rev. **10**, 207] that the generic hypersurface $F^{(n)}$ of S_r carries some S_k if $r\geqq\rho=k+\binom{n+k-1}{n-1}$, and it remained to show the existence of such S_k also if $k+(k+1)^{-1}\binom{n+k}{k}\leqq r<\rho$. The hypersurface $F^{(n)}$ can be obtained as the intersection of the generic hypersurface $\Phi^{(n)}$ of S_ρ with the generic S_r of S_ρ; so one shows that the $F^{(n)}$ so obtained contains some S_k. By the first note (which gave not only existence but also dimension), the S_k on $\Phi^{(n)}$ constitute an irreducible algebraic system of dimension $\theta=(n-1)\binom{n+k}{n}$. Of the $\infty^{(r+1)(\rho-r)}$ S_r of S_ρ, those which contain some S_k of $\Phi^{(n)}$ constitute an irreducible algebraic system Σ of dimension $(r+1)(\rho-r)-\eta$, $\eta\geqq 0$, and one has to show that $\eta=0$. One computes then that the general S_r of Σ contains an irreducible algebraic system of S_k on $\Phi^{(n)}$ of dimension $d=\delta+\eta$, where $\delta=(k+1)(r-k)-\binom{n+k}{k}$. Fixing an S_k^0 generically on $\Phi^{(n)}$, one constructs the correspondence associating a general S_k on $\Phi^{(n)}$ with the S_r of S_ρ containing S_k and S_k^0. Applying the principle of counting constants, one obtains $\eta=0$. In the application, one has to make use of the fact that two generic S_k on $\Phi^{(n)}$ are skew. This in turn depends on the following division of the S_k on $\Phi^{(n)}$ into special and nonspecial kinds: the S_{k+1} passing through an S_k on $\Phi^{(n)}$ cut $\Phi^{(n)}$ in this S_k and further in a V_k^{n-1} (possibly indeterminate in S_{k+1}) which either contains S_k as a component or cuts it in a V_{k-1}^{n-1}. The latter alternative is excluded if $r>\rho$ but obtains for nonspecial S_k if $r\leqq\rho$.

A. Seidenberg (Berkeley, Calif.).

Segre, Beniamino. Osservazioni sulle involuzioni piane più volte infinite. Boll. Un. Mat. Ital. (3) **3**, 196–200 (1948).

L'auteur définit une involution plane I_n^d de groupes de points G_n par récurrence sur d: $I_n^0=G_n$; I_n^1 est une série linéaire g_n^1 sur une courbe plane (sens restreint) ou même une involution irrationnelle γ_n^1 (sens large). Ensuite I_n^d est défini comme un système algébrique pur de G_n tel que les restes d'un point générique constituent une I_{n-1}^{d-2} [voir Severi, Serie, Sistemi d'Equivalenza, Cremonese, Rome, 1942; ces Rev. **10**, 206]. En considérant G_n comme une courbe de $n^{\text{ème}}$ classe, on représente la totalité des G_n par les points d'une W_{2n} de $S_{n(n+3)/2}$ et les I_n^d par certaines sous-variétés de W_{2n}; W_{2n} est rationnelle. L'auteur examine ensuite des exemples parmi lesquels une I_2^3 unirationnelle birationnellement équivalente à V_3^3 de S_4.

L. Gauthier (Nancy).

Vaccaro, Giuseppe. Sulle singolarità di una V_3^4 di S_4 con piano doppio. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 417–427 (1948).

La variété V_3^4 de S_4 à plan double a été étudiée par Marletta [Giorn. Mat. Battaglini **40**, 265–274 (1902); **41**, 47–61, 113–128 (1903)] mais l'étude exhaustive des singularités supplémentaires qu'elle peut contenir n'a pas été faite. L'auteur l'expose en application de ses résultats sur les V^n admettant des points $(n-2)$-uples [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 288–293, 314–321 (1947); ces Rev. **9**, 610]. Une V_3^4 de plan double π peut avoir au maximum 10 points doubles isolés. Si elle comporte un point triple dans π elle possède au maximum 9 points doubles coniques isolés. Si π est bihyperplanaire il peut y avoir au maximum 3 points triples dans π et il y a au plus 9 points doubles isolés. Si en un point double le cône tangent admet une droite double non incidente à π, il y a au maximum 6 points doubles isolés. La présence d'un point double unihyperplanaire entraîne l'existence de deux droites doubles. L'étude de certains cas où il y a une ou plusieurs lignes doubles est entreprise. L'auteur termine en montrant que si V_3^4 contient deux plans doubles incidents, elle admet au maximum quatre points doubles isolés.

L. Gauthier (Nancy).

Villa, M. Proprietà caratteristiche delle reti omaloidiche. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 122–127 (1948).

Villa, Mario. Proprietà caratteristiche delle reti omaloidiche. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 231–235 (1948).

La jacobienne J d'un réseau R se comporte régulièrement aux points-base de multiplicité s si elle y a la multiplicité éventuellement virtuelle $3s-1$ quelque soit l'ordre du voisinage du point multiple; cette définition posée une condition nécessaire et suffisante pour que R soit homaloïdique est que J se comporte régulièrement et ne coupe la courbe générique de R qu'aux points-base. Cette proposition résulte de l'égalité $\sum r(3r-1)x_r=3n(n-1)$ où x_r désigne le nombre de points d'ordre r, égalité qui condense les deux relations classiques de Cremona.

A la jacobienne J on associe la courbe J' formée de toutes les composantes de J prises chacune une seule fois; les courbes de R passant par un point de J ont une tangente fixe dite stationnaire; une condition nécessaire et suffisante pour que R soit homaloïdique est que cette tangente stationnaire soit tangente à J' ou indéterminée; la nécessité est immédiate, la suffisance se déduit de la première condition par l'intermédiaire de la série coupée sur J par les courbes d'un faisceau extrait de R. Cette propriété s'exprime sous une forme algébrique dont l'auteur pense que l'on puisse tirer une méthode algébrique de recherche des transforma-

tions de Cremona; si f, g, h sont 3 courbes engendrant R, et si J se comporte régulièrement l'on devra avoir

$$\left(f\frac{\partial g}{\partial x}-g\frac{\partial f}{\partial x}\right)\frac{\partial J'}{\partial y}-\left(f\frac{\partial g}{\partial y}-g\frac{\partial f}{\partial y}\right)\frac{\partial J'}{\partial x}\equiv 0 \pmod{J'}.$$

Si l'on prend J au lieu de J' la condition n'est que nécessaire.

Si dans R on considère un faisceau F individualisé par un point P de J, on appelle tangentes nodales celles à la courbe possédant un point double en P. De la deuxième condition on déduit que si R est homaloïdique une tangente nodale enveloppe J', l'autre enveloppe une courbe résiduelle. Cette propriéte se rapproche d'une propriété locale des correspondances ponctuelles au voisinage d'un point à Jacobien nul, déjà donnée par l'auteur et Julia.

L'auteur se propose de généraliser aux espaces supérieurs; peut-être dans cette voie se trouverait un procédé algébrique de recherche des transformations spatiales de Cremona plus puissant que le classique procédé arithmétique.

B. d'Orgeval (Grenoble).

Villa, Mario. Proprietà caratteristiche delle reti omaloidiche. Boll. Un. Mat. Ital. (3) **3**, 201–204 (1948).

L'auteur expose les résultats dont les démonstrations sont contenues dans une note donnée sous le même titre [voir l'analyse ci-dessus]. *B. d'Orgeval* (Grenoble).

Yang, Chung-Tao. A theorem on a field of rational normal curves. Univ. Nac. Tucumán. Revista A. **6**, 313–317 (1948).

Verfasser betrachtet das System aller rationalen Normkurven des komplexen P_n, die durch n feste Punkte eines Teil-P_{n-1} gehen, sodass 3 weitere Punkte des P_n in allgemeiner Lage eine Kurve des Systems festlegen. Er beweist dann folgenden Satz: Γ sei eine Kurve dieses Systems, und es werde eine durch eine zyklische Transformation der Periode 3 auf Γ bestimmte Dreierinvolution betrachtet. Es sei A, B, C eine Gruppe dieser Involution, $\bar{A}\subset\Gamma$ der vierte harmonische Punkt von A bezüglich B, C; P ein Punkt ausserhalb von Γ, sodass durch P, A, $\bar{A}$ wieder eine Kurve Γ des Systems bestimmt ist, auf der man den vierten harmonischen Punkt A' von P bezüglich A, $\bar{A}$ bestimmen kann. Dann ist der Ort von A', wenn A die Kurve Γ durchläuft, wieder eine Systemkurve, die durch die Doppelpunkte der Involution geht. Umgekehrt kann man von A' ausgehen und erhält als Ort für die Lage von A dann entweder Γ oder eine Reihe von $3^{n-1}-1$ weiteren Netzkurven, die alle durch die Doppelpunkte der Dreierinvolution auf Γ gehen.

W. Burau (Hamburg).

Differential Geometry

Mirguet, Jean. Convexité et double courbure des orthosurfaces. C. R. Acad. Sci. Paris **228**, 646–648 (1949).

The main result is as follows. "Sur une orthosurface à paratingent supérieur fini et à courbures bornées, la fermeture de l'ensemble des points à véritable double courbure est équivalente à l'ensemble des points où l'appui est non dense." The definitions of the French terms, for which English equivalents have not yet been introduced, are found in the author's previous paper [Revue Sci. **85**, 67–72 (1947); these Rev. **9**, 18]. *H. Busemann* (Los Angeles, Calif.).

Mirguet, Jean. La double courbure et les conditions géométriques intrinsèques de la dérivation seconde. Revue Sci. **86**, 323–328 (1948).

Let S be an orthosurface in E^3 [for the definition of this and other terms see the reference given in the preceding review] which has a continuous tangent plane in the neighborhood of p. With p as origin and the tangent plane of S at p as (x, y)-plane the tangent plane of S at a neighboring point $p'=(\Delta x, \Delta y, \Delta z)$ has the form

$$\Delta p(x-\Delta x)+\Delta q(y-\Delta y)-(z-\Delta z)=0$$

with $\Delta p\to 0$, $\Delta q\to 0$ for $p'\to p$. With $\Delta x=\Delta r\cos\varphi$, $\Delta y=\Delta r\sin\varphi$, it is assumed that the limits of sequences $\Delta p/\Delta r$ and $\Delta q/\Delta r$ for $\varphi\to\varphi_0$ and $r\to 0$ are finite. This limit process yields two contingents corresponding to $\Delta p/\Delta r$ and $\Delta q/\Delta r$, respectively. The point p is a point of "dérivation seconde" if these two contingents are planes. (This property is independent of the coordinate system.) The surface S is said to have "convergent intersections" at p if the intersection of the tangent planes to S at p' and p "tends" for $p'\to p$ to contain the center of p and p'. If S has convergent intersections at p then Euler's theorem for the distribution of the normal curvatures (in the weak sense of $\lim 2\Delta z/\Delta r^2$) is necessary and sufficient for p to be of "dérivation seconde." If S has at p "double courbure non-dégénérée" and $\lim 2\Delta z/\Delta r^2$ exists for $\varphi\to\varphi_0$ and $r\to 0$ for all φ_0 then S has convergent intersections at p and Euler's theorem holds.

H. Busemann (Los Angeles, Calif.).

Borisov, Yu. F. Curves on complete two-dimensional manifolds with boundaries. Doklady Akad. Nauk SSSR (N.S.) **64**, 9–12 (1949). (Russian)

The paper extends results of A. D. Alexsandrov for surfaces with bounded curvature without boundary and results of Cohn-Vossen for smooth surfaces with boundary to surfaces with bounded curvature which possess a boundary [for this and the following compare A. V. Aleksandrov, same Doklady (N.S.) **60**, 1483–1486 (1948); these Rev. **10**, 147]. The angle between two shortest connections (geodesics) issuing from one point always exists. The concept of total geodesic curvature of a curve towards a given side is generalized to include surfaces with boundary. A geodesic has nonpositive curvature towards either side. Let R be a geodesic polygon bounding a simply connected subregion G of the surface, with sides L_i, $i=1, \cdots, n$, and sector angle θ_i between L_i and L_{i+1} measured in G. The integral curvature $\omega(G)$ of G equals $2\pi-\sum_i\varphi(L_i)-\sum_i(\pi-\theta_i)$, where $\varphi(L_i)$ is the total geodesic curvature of L_i towards G. If R is, in particular, a geodesic triangle, $\varphi=\max_{i=1,2,3}\varphi(L_i)$, and the Euclidean triangle with sides L_i has angle θ_i' between L_i and L_{i+1}, then $-13\Omega-6\varphi\leqq\theta_i-\theta_i'\leqq 5\Omega$, where Ω is the absolute integral curvature of G. If the surface S is homeomorphic to a half-plane and possesses an integral curvature $\omega(S)$, and if the boundary curve possesses a total geodesic curvature φ then $\omega\leqq\pi-\varphi$. If, in addition, every subarc of a boundary is a shortest connection of its end points, then $\omega(S)\leqq-\varphi$. If S is homeomorphic to a strip bounded by parallel lines and if the two bounding curves have total geodesic curvatures φ_1, φ_2, then $\omega(S)\leqq-\varphi_1-\varphi_2$. No proofs are given.

H. Busemann (Los Angeles, Calif.).

*Bouligand, Georges. **Les Principes de l'Analyse Géométrique. Tome I. Leçons de Géométrie Vectorielle. Préliminaires à l'Étude de la Théorie d'Einstein.** 3d ed. Librairie Vuibert, Paris, 1949. x+436 pp.

This book is a detailed exposition of the algebra and calculus of vectors together with their applications to the differential geometry of curves and surfaces in ordinary space. Its level is suitable for advanced undergraduates or beginning graduate students of mathematics in American universities, and it should also be of interest to physicists and engineers. There are three main sections, whose contents are reasonably standard: I, Vectorial operations in linear geometry; II, Metric vectorial operations; III, Infinitesimal vectorial operations. The exposition emphasizes geometric and axiomatic aspects and minimizes purely formal matters. The last third of the book consists of six notes of a more advanced character: I, On the principles of tensor calculus; II, On Riemannian spaces of dimension greater than two; III, On the foundations of geometry; IV, On surfaces of constant total curvature and convex surfaces; V, Variants and extensions of vectorial methods (quaternions, Grassmann calculus, nonholonomic spaces, and minimal problems); and VI, Groups and differential geometries. *C. B. Allendoerfer* (Haverford, Pa.).

Giuliano, S. Sulle superficie che si corrispondono per ortogonalità di elementi lineari. Matematiche, Catania **3**, 34–39 (1948).

Let S and $\bar{S}$ be two surfaces in one-to-one point correspondence, corresponding points having the same parametric values (u, v), the equations of S and $\bar{S}$ being therefore $x^i=x^i(u, v)$, $\bar{x}^i=\bar{x}^i(u, v)$, $i=1, 2, 3$. Then S and $\bar{S}$ are said to correspond with orthogonality of linear elements if $\sum_{i=1}^{3}dx^i d\bar{x}^i=0$. The problem of solving this equation for $\bar{x}^i$ for particular surfaces S is discussed. *V. G. Grove.*

Rollero, Aldo. Punto doppio delle superficie. Matematiche, Catania **3**, 111–118 (1948).

The purpose of this paper is to write the power series expansion of a surface at a conical point, and to describe the tetrahedron of reference geometrically.

V. G. Grove (East Lansing, Mich.).

*Şemin, Ferruh. **Quelques remarques sur deux théorèmes classiques.** Université d'Istanbul. Faculté des Sciences. Recueil de mémoires commémorant la pose de la première pierre des Nouveaux Instituts de la Faculté des Sciences, pp. 45–46, Istanbul, 1948.

The two classical theorems are the formula of Beltrami-Enneper which gives the torsion of an asymptotic curve on a surface and the theorem that a linear line complex in space is determined by five projectively independent lines.

S. Chern (Chicago, Ill.).

Lips, L. Two cases in which there is a simple relation between a space curve and the edge of regression of its polar surface. Simon Stevin **26**, 104–128 (1949). (Dutch)

The edge of regression K_1 of the developable surface generated by the polar lines of a curve K_0 is the locus of the center of the osculating sphere to the curve K_0. The edge of regression of the polar surface of K_1 is called K_2. The author considers two special cases. (A) At corresponding points the curves K_0, K_1, and K_2 have the same first curvature ρ. It follows that the curves have the same second curvature τ also and that both curvatures are equal and proportional to $\sqrt{S}$ (S is the arc length of the curves K_n; $S_0=S_1=S_2=S$). Hence the curves are cylindrical helices and it is proved that the curves K_{2n} lie upon the same cylinder and the curves $K_0, K_1, K_2, \cdots$ together upon a helicoid. (B) The curvatures of K_0 and K_1 both satisfy the relation $\rho'\tau=\rho$. It follows that, for every K_n, $\rho_n=\tau_n=S_n$ and $S_2=2S_1=4S_0$. It is proved that the K_n lie upon the same cone of revolution. Several other properties are derived.

J. Haantjes (Leiden).

Lemoine, Simone. Recherche des directions principales virtuelles d'un élément linéaire donné. C. R. Acad. Sci. Paris **228**, 898–900 (1949).

Given a line element ds^2, the author shows that the determination of the principal directions of surfaces admitting this line element depends upon the integration of two partial differential equations of the third order in one unknown. Exceptions are developable surfaces and cylindrical molding surfaces. *C. B. Allendoerfer* (Haverford, Pa.).

Lalan, Victor. Sur le second indicatif d'un réseau asymptotique. C. R. Acad. Sci. Paris **228**, 900–902 (1949).

In contrast to the result of Lemoine [see the preceding review] it is shown that the determination of the asymptotic directions depends upon the integration of two partial differential equations of the second order with separated derivatives. *C. B. Allendoerfer* (Haverford, Pa.).

Lalan, V. Les surfaces envisagées dans leurs rapports avec leurs lignes minima. II. Bull. Soc. Math. France **76**, 20–48 (1948).

[The first part of this paper appeared in the same Bull. **75**, 63–88 (1947); these Rev. **9**, 375.] This part deals with the determination of a surface in terms of its minimal lines. Many of the results have been announced [C. R. Acad. Sci. Paris **223**, 883–885 (1946); **224**, 518–520, 1201–1203 (1947); these Rev. **8**, 350, 404, 530]. Let ω_1 and ω_2 be the minimal forms of the surface, H the mean curvature, and A the asphericity. The problem is to determine H and A by means of ω_1 and ω_2; for unless $A=0$, the knowledge of H and A gives expressions for the first and second fundamental forms of the surface.

The solution is obtained by considering the following Pfaffian system which is equivalent to the equations of Gauss and Codazzi:

$$\begin{aligned} dH&=-2A,\\ A^{-1}dA&=\chi-\rho\tilde{\omega},\\ d\rho&=\rho\chi-\Phi+(c-K/A)\tilde{\omega}/(rs), \end{aligned} \tag{1}$$

where χ, Φ, and $\tilde{\omega}$ are linear combinations of ω_1 and ω_2; ρ is an auxiliary variable; $d\omega_1=r[\omega_1\omega_2]$; $d\omega_2=s[\omega_2\omega_1]$; c is a function of r, s and their derivatives; and $K=H^2-A^2$ is the total curvature. The exterior differentiation of the system (1) gives the additional condition:

$$\rho d\chi=(\lambda+\mu H^2/A+\nu A)[\omega_1\omega_2], \tag{2}$$

where λ, μ, and ν are functions of ω_1, ω_2 and their derivatives. If $d\chi\neq 0$, it is possible to solve algebraically for H and A. If $d\chi=0$, this solution is still usually possible, but in an exceptional case it is necessary to integrate a first order differential equation. This gives ∞^1 essentially distinct solutions.

When there are two surfaces with the same ω_1 and ω_2, they are related conformally with the preservation of the pseudo-arcs of the minimal lines; i.e., there is a minimal conformal representation. It is proved that such pairs of surfaces

depend only on arbitrary functions of a single argument. When a surface is known to be a member of such a pair, the other surface can usually be determined uniquely. An exception is a surface of isothermal mean curvature, and a detailed study of this is carried out. The paper concludes with an elaborate treatment of other special cases.

C. B. Allendoerfer (Haverford, Pa.).

Maneng, Louis. Sur les familles de surfaces admettant les mêmes formes minima. C. R. Acad. Sci. Paris **228**, 805–806 (1949).

The equations of Gauss and Codazzi may be expressed as a Pfaffian system involving the minimal forms of a surface. If this system is completely integrable there exists a triple infinity of surfaces admitting the given minimal forms. The author shows that this happens only for surfaces of Weingarten on which the lines of constant mean curvature are isothermals and geodesic parallels. *C. B. Allendoerfer.*

***Bloch, André, et Guillaumin, Gustave. La Géométrie Intégrale du Contour Gauche.** Gauthier-Villars, Paris, 1949. vi+144 pp.

For a plane area bounded by a Jordan curve C, it is well-known that the area and the various moments can be expressed as contour integrals around C. For instance: $A=\frac{1}{2}\int x\,dy-y\,dx$; $M_y=\frac{1}{3}\int x^2\,dy-2xy\,dx$. The purpose of this book is to study the analogous integrals belonging to a skew unknotted space curve. Specifically this involves the properties of the integrals $I=\int_C x^\alpha y^\beta z^\gamma(\lambda\,dx+\mu\,dy+\nu\,dz)$, where α, β, and γ are nonnegative integers and λ, μ, ν are real numbers. There is no immediate connection between this material and the "integral geometry" of Blaschke and his associates, nor between it and other integral theorems involving the curvature of the curve.

The first important concept introduced is that of the vector area of C. This is the vector $\mathbf{A}=\frac{1}{2}\int\mathbf{P}\times d\mathbf{P}$, where $\mathbf{P}=(x, y, z)$. Let $\mathbf{n}$ be the unit normal to a surface Σ bounded by C and let $d\sigma$ be the element of area of Σ. Then also $\mathbf{A}=\int_\Sigma \mathbf{n}\,d\sigma$; i.e., it is the resultant of the system of vectors $\mathbf{n}\,d\sigma$. This concept is given various physical interpretations by assigning physical meanings to $\mathbf{P}$. A second concept is the moment of Guldin, $\mathbf{M}$, defined as the integral of the moments of the vectors $\mathbf{n}\,d\sigma$ about the origin. Thus $\mathbf{M}=\int_\Sigma \mathbf{P}\times\mathbf{n}\,d\sigma$, and its components can be expressed as contour integrals of the form: $M_x=-\frac{1}{2}\int(y^2+z^2)\,dx$. If C is displaced so that its infinitesimal translation is $d\rho$ and its infinitesimal rotation is $d\omega$, the volume V so enclosed is given by:

$$V=\mathbf{A}\cdot\int d\rho+\mathbf{M}\cdot\int d\omega.$$

This is the theorem of Koenigs.

An axis of gravity of C is defined as the locus of centers of gravity of the plane sections of a cylinder passing through C. The set of these axes forms the congruence of gravity of C. It is proved that this congruence contains three lines through each point of space and one line in each plane. Its focal surface is a developable of the third class tangent to the plane at infinity. Any congruence with these properties is a congruence of gravity. The characteristic tensors of C are defined by $\int_C \mathbf{P}^n\otimes d\mathbf{m}(\mathbf{P})$, where $\mathbf{P}^n$ is the n-fold tensor product of $\mathbf{P}\otimes\mathbf{P}\otimes\cdots\otimes\mathbf{P}$. When $n=2$ this gives the ordinary inertial tensor. The skew symmetric part of this tensor is formed from the components of $\mathbf{M}$; its symmetric part is called the orthaloid tensor. The properties of this tensor are developed. If $f(x, y, z)$ is an analytic function, $\int f ds$ can be expanded into a sum of integrals of the form I. Detailed computations for scalar and vector potentials and for the solid angle of C relative to a point are worked out and the terms of low order are interpreted in terms of the characteristic tensors.

The book closes with a chapter on the extension of these ideas to non-Euclidean spaces and with four supplementary notes, one of which is an extension of the main ideas to Euclidean spaces of n dimensions. *C. B. Allendoerfer.*

Tigano, O. Sulla determinazione delle curve di Mannheim. Matematiche, Catania **3**, 25–29 (1948).

If the curvature and torsion κ and τ of a curve C satisfy the equation $\kappa^2+\tau^2=\kappa/h$ (h=constant), C is said to be a curve of Mannheim. It is shown that the most general curve of Mannheim is the locus of the point (x, y, z) where

$$x=\int_{s_0}^{s}\cos u\cos v\,ds,\quad y=\int_{s_0}^{s}\sin u\cos v\,ds,\quad z=\int_{s_0}^{s}\sin v\,ds,$$

$u=u(s)$, $v=v(s)$, s being the arc length of C.

V. G. Grove (East Lansing, Mich.).

Tigano, O. Sopra una classe di superficie che corrispondono per ortogonalità di elementi lineari ad una superficie di rotazione. Matematiche, Catania **3**, 30–33 (1948).

Let the parametric curves u=constant, v=constant be the lines of curvature of a revolute S. It is shown that if Σ is a surface of translation defined by

$$\xi=A_1(u)+B_1(v),\quad \eta=A_2(u)+B_2(v),\quad \zeta=A_3(u)+B_3(v),$$

where A_i and B_i are arbitrary functions, then the surface C_1 generated by the point (x, y, z):

$$x=A_2B_3-A_3B_2-(A_2A_3-B_2B_3)+2\int(A_3A_2'du-B_3B_2'dv),$$

y, z being found by cyclic change of subscripts, corresponds with orthogonality of linear element to the revolute S.

V. G. Grove (East Lansing, Mich.).

Sauer, Robert. Projektive Transformationen des Darbouxschen Flächenkranzes. Arch. Math. **1**, 89–93 (1948).

The following is a condensation of the author's introduction. Jede infinitesimale Verbiegung einer Fläche definiert zwölf in bestimmter Weise miteinander verknüpfte Flächen. E. Salkowski hat diesen von Darboux eingeführten Flächenkranz im Rahmen der affinen Differentialgeometrie untersucht [Affine Differentialgeometrie, de Gruyter, Berlin-Leipzig, 1934, pp. 176–179]. Noch durchsichtiger werden die Zusammenhänge im weiteren Rahmen der projektiven Differentialgeometrie. Wir werden im folgenden Transformationsgruppen angeben, bei denen irgendeine Fläche des Kranzes einer beliebigen projektiven Abbildung unterworfen wird und der ganze Kranz wieder in einen Darbouxschen Flächenkranz übergeht. *P. Scherk* (Saskatoon, Sask.).

Fassina, María C. The Peterson surfaces. Revista Unión Mat. Argentina **13**, 172–182 (1948). (Spanish)

L'auteur passe en revue les propriétés essentielles des surfaces de Peterson relatives au réseau conjugué de Koenigs et à l'applicabilité d'une surface de Peterson donnée sur ∞^1 autres surfaces de la même espèce avec conservation des lignes de niveau et des profils méridiens. Elle rappelle ensuite un résultat de Paris sur les surfaces admettant deux axes de Peterson, lesquelles admettent aussi un troisième axe de Peterson formant avec les deux premiers un trièdre

trirectangle. Puis elle recherche la condition d'existence de surfaces de Peterson avec deux axes non concourants, à laquelle elle rattache le rèsultat particulier obtenu par Paris.

P. Vincensini (Besançon).

Terracini, A. Su alcuni sistemi triplamente infiniti di curve su una superficie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 545–547 (1947).

A system of ∞^3 curves on a surface is said to possess "the projective property" when at any two fixed points the pencils of tangents to the ∞^1 curves of the system through the points are in projective relationship. The author is interested in when this projective property subsists to various orders of approximation, a problem he has previously studied in the plane [Univ. Nac. Tucumán. Revista A. **6**, 273–287 (1947); these Rev. **10**, 538]. Analytic results there obtained are applied to the present case. First order approximation requires the curve family to satisfy a third order differential equation of specific form. Second order approximation imposes specific conditions on the coefficients in the equation. B. Segre's projective lines on a surface furnish an example of the latter.

J. L. Vanderslice.

Bell, P. O. Differential geometry of a general motion in the complex projective line. Amer. J. Math. **71**, 427–442 (1949).

The motions in the complex projective line are studied by methods of projective differential geometry. Two invariants play an important role, namely the real and imaginary parts of the Schwarzian derivative. The motions for which these invariants are constant are proved to be the ones which admit homographies into themselves. Interpretations of the invariants are given.

S. Chern (Chicago, Ill.).

Dalla Volta, Vittorio. Sull'isometria di calotte superficiali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 381–384 (1948).

This paper is a report on results which are to be published in detail elsewhere. The main result, a generalization of one by Bompiani, is the following. A necessary and sufficient condition for existence of an isometric correspondence of order $h \geqq 3$ between two surface calottes of order $n \geqq h-1$ can always be reduced to the equality of the values at the centers of the calottes of the total curvatures of the two calottes and of their derivatives up to order $h-3$ inclusive.

J. L. Vanderslice (College Park, Md.).

Longo, C. Invarianti proiettivi di calotte del 3° ordine tangenti in un punto. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 295–326 (1948).

This paper gives an exhaustive study of the projective properties of the set of surface elements (or caps) of the third order σ_3 having the same center and the same tangent plane, in the ordinary projective space S_3.

The following cases may be distinguished. (1) The set of caps σ_3 having a cap σ_2 of the second order in common: these caps can be represented as points of a projective space R_4 with three hyperplanes R_3 representing exceptional caps. A collineation in S_3 of the set of caps induces a collineation in R_4 leaving the three hyperplanes invariant. Two caps determine a pencil of caps which can be defined in S_3 by the use of appropriate cubic monoids; a cap in a given pencil has one invariant.

(2) The set of caps σ_3 having the same asymptotic tangents (but not the same cap σ_2). These caps can be thought of as points in a projective space R_5, with four exceptional hyperplanes R_4. A collineation in S_3 induces a quadratic transformation in R_5 whose homaloidal system is completely characterized with respect to the exceptional hyperplanes. The notion of pencil of caps holds also in this case; the invariant of a pencil and the three invariants of two caps are also determined.

(3) The set of caps with one asymptotic tangent in common represented on an R_6, whose exceptional elements are three R_5 and an R_4 in one of them and a cone V_3^4; a collineation in S_3 induces a quadratic transformation whose homaloidal system is determined in relation to the exceptional elements. Two caps have four invariants which are also determined.

(4) The set of all caps having the same center and tangent plane (but no other elements in common) represented on an R_7 with the following exceptional elements: an R_6, a quadric cone with an R_4 as vertex, a manifold V_6^3. A collineation in S_3 induces a quadratic transformation in R_7; its homaloidal system and the five invariants of two caps are determined (as well as their interpretation in S_3).

E. Bompiani (Rome).

Ascoli, Guido. Sopra una proprietà delle normali ad una superficie ed una sua parziale estensione. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 280–285 (1948).

Let C be a congruence of lines $l(p)$ issuing from the points p of a surface S. On every $l(p)$ lay off from p (in both directions) a segment of constant length. Under bending of S change each $l(p)$ so that it keeps the same angle with S at p. The paper studies the question: which congruences have the property (E) that for every sufficiently small h and every subsurface of S the volume of the set of these segments is invariant under bending of S? (Tricomi proved that the normal congruences have this property.) Either of the following conditions is necessary and sufficient. (1) Bending of S induces an area-preserving mapping for the spherical image of C. (2) If the versor of C at p is projected on the tangent plane of S at p, then the resulting vector field on S is the vector field associated with an infinitesimal (rigid) transformation of S into itself. It follows that for general surfaces the normal congruences are the only congruences with property E.

H. Busemann (Los Angeles, Calif.).

***Kuiper, Nicolaas Hendrik. Onderzoekingen over Lijnenmeetkunde. [Investigations on Line Geometry].** Thesis, University of Leiden, 1946. iv+60 pp. (Dutch. French and English summaries)

This thesis gives the background of other work by the same author [Nederl. Akad. Wetensch., Proc. **51**, 1137–1145, 1244–1250 = Indagationes Math. **10**, 361–369, 388–394 (1948); these Rev. **10**, 326, 403]. Using the Study method of dual vectors, the author develops the line geometry in a Euclidean three-space (without using point coordinates). This enables him to state a "translation rule," which converts theorems in spherical or two-dimensional elliptic geometry to line geometry theorems in Euclidean three-dimensional space. The results (known) about infinitesimal properties of surfaces are used in the generalization of Bertrand curves. The author finds rectilinear surfaces with identical principal normals (the second axis of the canonical trihedron). Homological lines on such surfaces have an arbitrary constant dual angle. Finally, the author uses his

method in order to build up the background for investigations by dual vectors of two- and three-dimensional line spaces (theory of line congruences, line complexes and line congruences in line complexes). *V. Hlavatý.*

Maeda, Kazuhiko. Characteristic properties of the involute of a circle. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **32**, 133–147 (1945).

Various properties of some special curves in the Laguerre plane are established. The following example seems typical. The satellite curve of an equiangular spiral is again an equiangular spiral with the same pole and angle [cf. Maeda, Jap. J. Math. **18**, 385–581 (1942); these Rev. **8**, 232]. *P. Scherk* (Saskatoon, Sask.).

Dinghas, Alexander. Zur Metrik nichteuklidischer Räume. Math. Nachr. **1**, 287–291 (1948).

Considerations along classical lines of Riemannian spaces with constant curvature. *C. Y. Pauc* (Cape Town).

Taub, A. H. A characterization of conformally flat spaces. Bull. Amer. Math. Soc. **55**, 85–89 (1949).

The author's purpose is to display some analogies between the theory of groups of infinitesimal conformal transformations and that of groups of motions in Riemannian spaces. He shows: (1) that, if $\xi_{\sigma|i}$ ($\sigma=1, 2, \cdots, r$) are components of infinitesimal conformal transformations of a Riemannian V_n, so are $a^\sigma\xi_{\sigma|i}$, where the a's are arbitrary constants; (2) that, if $X_\sigma f=\xi_\sigma{}^i\partial f/\partial x^i$ are the generators of infinitesimal conformal transformations of a V_n, so are the commutators $(X_\sigma, X_\tau)f$ [cf. Eisenhart, Riemannian Geometry, Princeton University Press, 1926, p. 237]; and (3) that a necessary and sufficient condition for a V_n ($n\geqq 3$) to admit a $\frac{1}{2}(n+1)(n+2)$-parameter group of infinitesimal conformal transformations is that the V_n be conformal to a flat space. *H. S. Ruse* (Leeds).

Ruse, H. S. Three-dimensional spaces of recurrent curvature. Proc. London Math. Soc. (2) **50**, 438–446 (1949).

A V_n for which $\nabla_\mu K_{\nu\lambda\kappa}=\kappa_\mu K_{\nu\lambda\kappa}$ is called a space of recurrent curvature or K_n [this is not the K_n of Schouten and Struik, Einführung in die neueren Methoden der Differentialgeometrie, v. 2, 2d ed., Noordhoff, Groningen-Batavia, 1938, pp. 232 ff., but what hitherto was called a κ-space]. This paper deals with the case $n=3$. The main conclusion is that a necessary and sufficient condition for a V_3 to be a K_3 is that it should admit a field of parallel vectors; κ_i is the gradient of a scalar that is equal to $\log|K|$ if $K\neq 0$. If r^ν is the field of parallel vectors, $K_{\nu\lambda\kappa}$ can be expressed in terms of $r^\nu I_{\lambda\kappa\mu}$. If $r^\nu r_\nu\neq 0$, the K_3 is the product of a K_2 and an R_1. If and only if $r^\nu r_\nu=0$, K vanishes. The linear element then has the form $a_{\alpha\beta}dx^\alpha dx^\beta+2dx^3dx^3$; $\alpha, \beta=1, 2$. Some of the results are due to A. G. Walker, whose paper on the geometry of the K_n is in preparation. At the end of the paper there is a remark on simply harmonic K_4's treated in a former paper. *J. A. Schouten* (Epe).

Lichnerowicz, André. Quelques théorèmes de géométrie différentielle globale. Comment. Math. Helv. **22**, 271–301 (1949).

The paper is a detailed account of results announced earlier [C. R. Acad. Sci. Paris **223**, 12–14 (1946); **227**, 711–712 (1948); these Rev. **8**, 490; **10**, 203]. It is divided into four parts. Part I is concerned with relations between operations on differential forms in the base space B and the total space E of a fibre bundle, both having a Riemannian metric. Let $*$ be the adjoint operator, d the exterior differentiation, $d^*=*d*$, and p^{-1} the inverse mapping of differential forms of B into those of E. A formula is given for the operation $d^*p^{-1}-p^{-1}d^*$ in E, from which geometrical conclusions are drawn. Part II studies, for the case of compact fibres, a mapping of differential forms of E into forms of the same degree of B. It is defined by taking a form Φ of degree equal to the dimension of the fibres, transversal to all fibres, multiplying a form Ω of E by Φ, and integrating over the fibres. If Φ is closed and the field defined by it is completely integrable, with $\int_{F_n}\Phi\neq 0$, F_n being a fibre, then the mapping defines a homomorphism of the cohomology group $H_p(E)$ into the cohomology group $H_p(B)$. Part III draws conclusions from the general considerations of the first two parts. Typical among these are the inequalities $b_p(B)\leqq b_p(E)$ (b_p being the Betti number of dimension p), which hold under different hypotheses. The fibre bundles whose fibres are the Stiefel manifolds formed by the tangent vectors are also studied. Among the results obtained we quote the following. The space of unit tangent vectors to a locally flat (in the sense of the Riemannian metric) compact orientable manifold has the same Betti numbers as that of $B\times S_{n-1}$, where S_{n-1} is the $(n-1)$-sphere. Part IV is concerned with the extension of the Gauss-Bonnet formula for Riemannian manifolds to a class of Finsler manifolds, called the Berwald manifolds. As a corollary, no Minkowski metric of zero scalar curvature, everywhere regular, can be given on an even-dimensional sphere. *S. Chern* (Chicago, Ill.).

Bochner, S. Curvature and Betti numbers. II. Ann. of Math. (2) **50**, 77–93 (1949).

The author generalizes his previous work on connections between various curvatures and Betti numbers of compact Riemannian manifolds [see Ann. of Math. (2) **49**, 379–390 (1948); these Rev. **9**, 618] in two directions: (1) to a compact differentiable manifold M_n with a general (not necessarily symmetric) affine connection, (2) to a compact complex Hermitian manifold V_{2k} [see Bull. Amer. Math. Soc. **53**, 179–195 (1947); these Rev. **8**, 490]. In the affine case the results are in terms of B_p, the number of linearly independent "pseudoharmonic" tensors of order p (B_p is not known to be equal to the pth Betti number of M_n); inequalities on B_p are obtained if M_n is assumed flat, or if the "pseudo-Ricci" tensor is assumed positive. In the Hermitian case, it is proved that if the Ricci tensor of V_{2k} is positive and if V_{2k} is conformally flat (or "nearly" so), then the Betti number B_{2l+1} is zero and B_{2l} equals 1, for $0\leqq l\leqq k$. Here "conformal flatness" is in terms of any one of several formal analogues of the usual conformal curvature tensor in the Riemannian case. Complex projective space P_{2k} is the type example of a conformally flat V_{2k} with positive definite Ricci tensor. *S. B. Myers* (Ann Arbor, Mich.).

Pickert, Günter. Elementare Behandlung des Helmholtzschen Raumproblems. Math. Ann. **120**, 492–501 (1949).

Refinement, by elementary means, of the conditions under which metric Pythagorean geometry can be characterized in the affine vector geometry. The results are compared with those obtained by Iyanaga and Abe [Proc. Imp. Acad. Tokyo **19**, 174–180, 540–543 (1943); these Rev. **7**, 237]. *E. Bompiani* (Rome.).

Petrov, P. I. Differential invariants of generalized spaces with symmetrical affine connection. Doklady Akad. Nauk SSSR (N.S.) 65, 129–130 (1949). (Russian)

Since the curvature tensor $B^i{}_{jkl}$ of an S_2 with symmetric affine connection $\Gamma^i{}_{jk}$ is expressible in terms of a tensor ψ_{jk}, any affine differential invariant of the second order in $\Gamma^i{}_{jk}$ is expressible in terms of ψ_{jk} and its covariant derivative $\psi_{jk,l}$. By contraction with the invariant tensor density e^{ijk}, a complete system of nine functionally independent invariants is formed from the base tensors ψ_{jk} and $\psi_{jk,l}$. The conditions that S_2 be projectively flat, heretofore expressed by a vector equation, are now expressible as a pair of conditions among the basic invariants. *T. C. Doyle.*

* **Nazim Terzioğlu, A. Über den Satz von Gauss-Bonnet im Finslerschen Raum.** Université d'Istanbul. Faculté des Sciences. Recueil de mémoires commémorant la pose de la première pierre des Nouveaux Instituts de la Faculté des Sciences, pp. 26–32, Istanbul, 1948.

The Gauss-Bonnet formula on a surface in three-dimensional Finsler space is established. The author imbeds the boundary curve into a family of curves, thus reducing the problem to one of Riemannian geometry. *S. Chern.*

Golab, S. Alcuni teoremi della teoria degli oggetti geometrici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 5, 120–122 (1948).

In this note the author gives a survey of the results which he has obtained concerning the problem of finding all the geometric objects with k components of a certain class v in an n-dimensional space X_n. An object is said to be a differential geometric object if the functions which give the transformation of the components do not involve the coordinates of the point explicitly. The problem mentioned above has been completely solved for differential geometric objects with one component. The solution for $k \neq 1$, $v=1$, $n=1$ is announced. [For the details of the results we may refer to the following notes: Gołąb, Math. Z. 44, 104–114 (1938); Ann. Soc. Polon. Math. 19 (1946), 7–35 (1947); 20 (1947), 10–27 (1948); Pensov, C. R. (Doklady) Acad. Sci. URSS (N.S.) 54, 563–566 (1946); these Rev. 9, 206; 10, 267; 9, 67.] *J. Haantjes* (Leiden).

Padma, N. On the duality of linear tensors in affine space. J. Indian Math. Soc. (N.S.) 12, 107–116 (1948).

The author uses the Schouten n-vector densities $\epsilon_{\lambda_1 \cdots \lambda_n}$, $\epsilon^{\nu_1 \cdots \nu_n}$ [Schouten and Struik, Einführung in die neueren Methoden der Differentialgeometrie, v. 1, 2d ed., Noordhoff, Groningen, 1935, p. 29; the book seems to be unknown to the author] to introduce the mutually correspondent m- and $(n-m)$-vector densities [loc. cit., p. 160], and to solve some elementary questions about their covariant derivatives [cf., for instance, loc. cit., p. 82, problems 7, 10]. *V. Hlavatý* (Bloomington, Ind.).

* **Grammel, R. A simple representation of tensors and affinors (dyadics).** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 394–403. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

The author offers a geometric interpretation of the second order tensor by means of the following. Let Φ denote a dyad; let i, j and t, n denote two sets of orthogonal unit vectors; let X_x, X_y, Y_x, Y_y and N_n, N_t, T_n, T_t denote the components of the dyad (tensor) with respect to these two sets of unit vectors. If ϕ denotes the angle between i and n then

$$N_n = X_x \cos^2\phi + Y_y \sin^2\phi + (X_y + Y_x)\sin\phi\cos\phi,$$

etc. For fixed X_x, Y_y, X_y, Y_x, and variable ϕ, the above equation leads to a polar coordinate representation of N_n. Figures are drawn for N_n, N_t. *N. Coburn.*

Castoldi, Luigi. Appunti per una teoria intrinseca delle variabili complesse sopra una superficie. Pont. Acad. Sci. Acta 11, 259–271 (1947).

Let g_{ij} be the metric tensor of a surface, ϵ^{ij} its Ricci-tensor and $a=a(x^1, x^2)$, $b=b(x^1, x^2)$ two scalar functions. If we prescribe for the tensor (1) $T_j{}^i = a\delta_j{}^i + b\epsilon^{ih}g_{hj}$ the condition (2) $\nabla_i T_j{}^i = 0$ we get a condition which for a plane reduces to (3) $\partial_1 a = \partial_2 b$, $\partial_2 a = -\partial_1 b$. The equations (3) have the form of the Cauchy-Riemann condition. For that reason the author calls $T_j{}^i$ defined by (1), (2) "a complex variable" on a surface and investigates other analogies with the theory of complex variable (analytic functions of $T_j{}^i$, the Cauchy identity $\int f(z)dz = 0$, and so on). *V. Hlavatý.*

NUMERICAL AND GRAPHICAL METHODS

* **Tables of Generalized Sine- and Cosine-Integral Functions. Part I. By the Staff of the Computation Laboratory.** Harvard University Press, Cambridge, Mass., 1949. xxxviii+462 pp. $10.00.

* **Tables of Generalized Sine- and Cosine-Integral Functions. Part II. By the Staff of the Computation Laboratory.** Harvard University Press, Cambridge, Mass., 1949. viii+560 pp. $10.00.

These two volumes give values of the six functions

$$S(a,x) = \int_0^x \frac{\sin(a^2+x^2)^{\frac{1}{2}}}{(a^2+x^2)^{\frac{1}{2}}}dx, \quad C(a,x) = \int_0^x \frac{1-\cos(a^2+x^2)^{\frac{1}{2}}}{(a^2+x^2)^{\frac{1}{2}}}dx,$$

$$Ss(a,x) = \int_0^x \frac{\sin(a^2+x^2)^{\frac{1}{2}}\cdot\sin x}{(a^2+x^2)^{\frac{1}{2}}}dx,$$

$$Sc(a,x) = \int_0^x \frac{\sin(a^2+x^2)^{\frac{1}{2}}\cdot\cos x}{(a^2+x^2)^{\frac{1}{2}}}dx,$$

$$Cs(ax) = \int_0^x \frac{\cos(a^2+x^2)^{\frac{1}{2}}\cdot\sin x}{(a^2+x^2)^{\frac{1}{2}}}dx,$$

$$Cc(a,x) = \int_0^x \frac{\cos(a^2+x^2)^{\frac{1}{2}}\cdot(1-\cos x)}{(a^2+x^2)^{\frac{1}{2}}}dx.$$

Values are given to 6 decimals for the following combinations of a and x:

a	x
0(0.01)1	0(0.01)0.99
0(0.02)2	0(0.02)1.98
0(0.05)5	0(0.02)2(0.05)4.95
0(0.1)10	0(0.1)9.9
0(0.2)25	0(0.2)25.

Thus, for $a=0.97$, $x=0(0.01)0.99$;
while for $a=0.95$, $x=0(0.01)1(0.02)2(0.05)4.95$;
for $a=1.4$, $x=0(0.02)2(0.05)5(0.1)10(0.2)25$;
for $a=1.5$, $x=0(0.02)2(0.05)5(0.1)10$.

The arrangement is such that tables for each value of a are completed before the next is started (inconvenient for a-wise interpolation). Part I contains values for $a<2$, and part II values for $a \geqq 2$.

A useful introduction outlines the properties of the functions, method of computation, methods of interpolation and has a short section on applications, with a number of references to the literature of the subject. Some connections with more familiar functions, given in the introduction, seem worth quoting here:

$$S(0,x)=2Sc(0,\tfrac{1}{2}x)=2Cs(0,\tfrac{1}{2}x)=\mathrm{Si}\,(x),$$
$$C(0,x)=2Ss(0,\tfrac{1}{2}x)=2Cc(0,\tfrac{1}{2}x)+2C(0,\tfrac{1}{2}x)$$
$$=\gamma+\log x-\mathrm{Ci}\,(x),$$

where Si (x) and Ci (x) are the well-known sine and cosine integrals. Again,

$$Sc(a,x)=\tfrac{1}{2}[\mathrm{Si}\,(z)-\mathrm{Si}\,(y)],$$
$$Cs(a,x)=\tfrac{1}{2}[\mathrm{Si}\,(z)+\mathrm{Si}\,(y)]-\mathrm{Si}\,(a),$$
$$Ss(a,x)=\tfrac{1}{2}[\mathrm{Ci}\,(z)+\mathrm{Ci}\,(y)]+\mathrm{li}\,(a),$$
$$\overline{Cc}(a,x)=\int_0^x\frac{\cos(a^2+x^2)^{\frac{1}{2}}\cdot\cos x}{(a^2+x^2)^{\frac{1}{2}}}dx$$
$$=\sinh^{-1}x/a-C(a,x)-Cc(a,x)$$
$$=\tfrac{1}{2}[\mathrm{Ci}\,(z)-\mathrm{Ci}\,(y)],$$

where $z=(a^2+x^2)^{\frac{1}{2}}+x$, $y=(a^2+x^2)^{\frac{1}{2}}-x$.

J. C. P. Miller (London).

***Tables of Inverse Hyperbolic Functions. By the Staff of the Computation Laboratory.** Harvard University Press, Cambridge, Mass., 1949. xx+290 pp. $10.00.

These tables fill a noticeable gap. They give 4 decimal values as follows:

table I: $\tanh^{-1}x$, $x=0$ to 0.99999,
table II: $\sinh^{-1}x$, $x=0(0.002)3(0.005)3.5$,
table III: $\cosh^{-1}x$, $x=1$ to 3.5,
table IV: $\sinh^{-1}x$ and $\cosh^{-1}x$, $x=3.5$ to 22980.

The interval of tabulation varies from 0.001 to 0.00001 in table I, and from 0.00001 to 20 in tables III and IV. The first and second differences provided are adequate for interpolation everywhere, except when $x>0.9988$ for $\tanh^{-1}x$, and when $x<1.0004$ for $\cosh^{-1}x$, provided Everett's interpolation formula is used; these limits must be extended if third differences are not taken into account. The introduction indicates the errors of linear and quadratic interpolation, and gives alternative ways for obtaining the required functions in difficult regions. *J. C. P. Miller* (London).

Malavard, Lucien, et Tissot, Jean. Sur une méthode utilisant le bassin électrique pour la détermination des racines d'une équation algébrique. C. R. Acad. Sci. Paris **227**, 620–622 (1948).

The authors first make four observations based on their practical experience at the Laboratoire Analogique de l'Office National d'Études et de Récherches Aéronautiques where they have applied the method of F. Lucas [same C. R. **106**, 645–648 (1888)] to equations of degree as high as nine with real coefficients. This method consists in using a conducting sheet (electrolytic tank) as z-plane and introducing currents of predetermined intensity μ_k at arbitrarily but permanently selected points λ_k of the axis of reals. To avoid the difficulties and inaccuracies inherent in this method the intervals between the points λ_k are materialized as electrodes instead and they are placed at potentials of which the successive differences are the μ_k. The nodes of the equipotential lines then give the roots. Details of physically locating these are not given. The undesirable effects of the finite size of the z-plane are avoided by using an inverse Joukowski transformation such that the part of the axis of reals of the z-plane which would be occupied by electrodes corresponds to a semicircle in a Z-plane. Even when the roots are widely spaced they are said to be determined with relative error varying from .01 to .001. *R. Church.*

Best, G. C. Notes on the Graeffe method of root squaring. Amer. Math. Monthly **56**, 91–94 (1949).

In order to obtain a better idea of the multiplicity of equal moduli of the roots of an algebraic equation when employing Graeffe's method, the author constructs an auxiliary table of the quotients $\rho_m=a_{m-1}^2/a_m$, where a_m is the coefficient in the mth transformed equation which arises from a_{m-1} by Graeffe's method. In the reviewer's opinion, this decision can be made more simply by selecting only those m for which $a_m=a_{m-1}^2$ (these m are obtained automatically) and splitting the transformed equation into a set of equations M_i each of which has roots of equal modulus. [See Bodewig, Quart. Appl. Math. **4**, 177–190 (1946), in particular, pp. 179–181; these Rev. **8**, 53.] *E. Bodewig* (The Hague).

Bodewig. Über die Methode von Graeffe. Z. Angew. Math. Mech. **29**, 91–93 (1949).

Two modifications are given for the computation of the arguments of the roots of an algebraic equation by Graeffe's method. The first modification was given by Graeffe, the second is based on a method of Carvallo and the author [Bodewig, Quart. Appl. Math. **4**, 177–190 (1946); these Rev. **8**, 53]. *E. Frank* (Chicago, Ill.).

Bodewig, E. Konvergenztypen und das Verhalten von Approximationen in der Nähe einer mehrfachen Wurzel einer Gleichung. Z. Angew. Math. Mech. **29**, 44–51 (1949). (German. Russian summary)

This paper is concerned with the approximation of the roots of an equation. A measure of the rapidity of the convergence of a sequence is established and "the degree of convergence" is defined. It is thereby possible to close two "gaps" in the theory of approximation methods, namely, the comparison of the efficiency of various methods, and the dependence of the convergence on the multiplicity of the roots. It is proved that the convergence of many methods depends on the multiplicity, for example, in Newton's method and a generalization thereof which is given here, in Laguerre's method, and in a general method which furnishes a convergence of degree n. *E. Frank* (Chicago, Ill.).

Bertiau, F. General method for the approximate solution of equations and systems with two or more unknowns. Simon Stevin **26**, 90–103 (1949). (Dutch)

An equation in one unknown is brought into the form $x=g(x)$ with $g'(x)$ as small as possible. Then (a) for $|g'(x)|<0.2$ use simple iteration $x_{i+1}=g(x_i)$; (b) for $-1.2<g'(x)<-0.8$ put $x_{i+2}=(x_i+x_{i+1})/2$, where $x_{i+1}=g(x_i)$; (c) in all other cases put $x_{i+3}=(x_ix_{i+2}-x_{i+1}^2)/(x_i-2x_{i+1}+x_{i+2})$ [this formula was not originated by Lemaître in 1942, as the author states, but by Steffensen, Skand. Aktuarietidskr. **16**, 64–72 (1933)]. When dealing with two equations $x=\varphi(x,y)$, $y=\psi(x,y)$: (a) use the tangent planes instead of the surfaces, thus obtaining the usual formulas, among which are Newton's formula; (b) for $\varphi_x'\approx\psi_y'\approx-1$ and $|\varphi_y'|<1$, $|\psi_y'|<1$ put $x_{n+1}=(x_n+\varphi(x_n,y_n))/2$, $y_{n+1}=(y_n+\psi(x_n,y_n))/2$; (c) for $\varphi_x'\approx0$ or $\psi_y'\approx0$ put $x_{n+1}=(x_n,y_n)$, $y_{n+1}=(x_n,y_n)$. When dealing with more equations replace the surfaces by

their tangent planes. Of course, in all these procedures a first approximation must be known. *E. Bodewig.*

Hartree, D. R. Notes on iterative processes. Proc. Cambridge Philos. Soc. **45**, 230–236 (1949).
Domb, C. On iterative solutions of algebraic equations. Proc. Cambridge Philos. Soc. **45**, 237–240 (1949).

The first author indicates a classification of iterative processes. [This is not new: it was given in the same form by Schröder, Math. Ann. **2**, 317–365 (1870), pp. 330 ff. The author's questions, such as the nonuniqueness of the iterative processes and the developments still required in order to obtain all processes of a certain order, were answered by Schröder.]

The second author gives an application of the "Hartree processes" to algebraic equations. *E. Bodewig.*

Fox, L. A short account of relaxation methods. Quart. J. Mech. Appl. Math. **1**, 253–280 (1948).

The author gives a short explanation of relaxation methods for solving linear equations and all problems which can approximately be reduced to them. The reviewer does not agree with many of the author's claims, for example, that the "relaxation method is the most recent . . . and the most powerful" one for solving linear equations. It is actually the method of Seidel [Abh. Akad. Wiss. München. Kl. 2. **1874**, no. 3, pp. 81–108] and von Mises and Pollaczek-Geiringer even gave the statical meaning of the method [Z. Angew. Math. Mech. **9**, 58–77, 152–164 (1929)]. The first condition for convergence of the method was given by von Mises and Trefftz in 1929 [cf. the reference above]. The assertion on page 253 that "any set of simultaneous linear equations can be solved by relaxation" contradicts the remark on page 258 that the process does not converge (in most cases). *E. Bodewig* (The Hague).

Müller, M. Über ein Eulersches Verfahren zur Wurzelberechnung. Math. Z. **51**, 474–496 (1948).

The author studies the system of k linear difference equations:

$$a_{i,n+1}=pr(a_{1n}+\cdots+a_{i-1,n})+a_{in}+p(a_{i+1,n}+\cdots+a_{k,n}),$$

where $i=1, 2, \cdots, k$; p is an arbitrary positive number; r is positive. Euler [Opera Omnia, ser. I, v. 6, pp. 240–262] studied the cases $k=2, 3, 4$ with $p=1$ and found

$$\lim a_{in}/a_{1n}=\rho^{i-1},$$

where $\rho=r^{1/k}$, $i=2, \cdots, k$. Krafft [Monatsh. Math. Phys. **49**, 312–315 (1941); these Rev. **4**, 90] generalised the theorem for every k if $p=1$ and all $a_{i0}\geqq 0$. The author now proves that Euler's theorem holds for every $p>0$ if and only if one of the two conditions is satisfied: either (a) $\sum_{i=1}^{k}a_{i0}/\rho^i\neq 0$ or (b) k is even, $p\neq A=(1+\rho)/(r+\rho)$, $a_{i0}=(-1)^i(1/k)\rho^iC$, where $C\neq 0$ is arbitrary. In case (a) the convergence is linear and the rate of the convergence will improve when p is chosen near A. In all other cases the process will diverge or not be applicable. *E. Bodewig.*

***Geiringer, Hilda. On the solution of systems of linear equations by certain iteration methods.** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 365–393. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

The chief result of the paper is that the well-known sufficient criterion for the convergence of the usual iteration is also sufficient for that of the Seidelian iteration. That is, "the Seidelian iteration will converge for any starting vector x_0 and for any order of the equations toward the solution of the system $Ax+r=0$ (with all $a_{ii}=1$) if every sum of the absolute values of the elements of a row (or column) is less than 2." This condition may even be weakened provided that the matrix A is "irreducible." For then all sums but one may even be equal to 2 and only one sum must be less than 2. Then both the usual and the Seidelian iterations will still converge for any x_0 and any order of the equations. [The author's conjecture that, whenever the usual iteration converges, the same will hold for the Seidelian iteration, has been refuted by Collatz, Z. Angew. Math. Mech. **22**, 357–361 (1942); these Rev. **5**, 50; and Stein and Rosenberg, J. London Math. Soc. **23**, 111–118 (1948); these Rev. **10**, 485.] The author further discusses the influence of the starting vector x_0 and of the order of the equations, and the acceleration of convergence by matrix squaring.

E. Bodewig (The Hague).

Roma, Maria Sofia. Il metodo dell'ortogonalizzazione per la risoluzione numerica dei sistemi di equazioni algebriche. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **189**, 12 pp. (1947).

[Reprinted from Rivista del Catasto e dei Servizi Tecnici Erariali, no. 1 (1946).] Two ways of orthogonalizing a set of linear equations are demonstrated and exemplified. The results of the second differ only by constant factors from those of the first, which is due to J. P. Gram [J. Reine Angew. Math. **94**, 41–73 (1883)]; also cf. E. Schmidt [Rend. Circ. Math. Palermo **25**, 53–77 (1908), in particular, p. 61]. No adequate discussion of advantages and disadvantages is given. *J. W. Tukey* (Princeton, N. J.).

***Hotelling, Harold. Practical problems of matrix calculation.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 275–293. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

The author describes three chief problems of matrix calculation and their mechanical devices: (a) multiplication, (b) inversion of matrices and solution of linear equations, (c) determination of latent roots and vectors. The results are well known. For the iteration process for the inverse of the matrix A: $C_{m+1}=C_m(2-AC_m)$ the author finds for the error of C_m the inequality $N(C_m-A^{-1})\leqq N(C_0)k^{2m}/(1-k)$, where $1>k=N(1-AC_0)$ and N is the norm. The author's remark about the amount of work involved in the computation of A^{-1} by partitioning A into submatrices and proceeding by Schur's relation [see the review of Guttman, Ann. Math. Statistics **17**, 336–343 (1946); these Rev. **8**, 171] can be completed in the sense that the amount of work is indeed independent of the manner of partitioning, as the reviewer proved [Nederl. Akad. Wetensch., Proc. **50**, 49–57 (1947); these Rev. **8**, 407]. At the end the author discusses the question of tables for A^{-1}. *E. Bodewig* (The Hague).

Grossman, D. P. The application of punched-card machines to the solution of a system of linear algebraic equations by the iteration method. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1229–1238 (1948). (Russian)

When written in matrix form the system of $2n$ equations in question is

$$(1)\qquad \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} X=M,$$

where A_{11} and A_{22} are real diagonal matrices. The real n by n matrices A_{12} and A_{21}, otherwise unrestricted in generality, remain fixed for a number of the aerodynamical problems which give rise to the system (1) while A_{11}, A_{22}, and M change from problem to problem. When the Gauss-Seidel method is applied to (1) it is seen to involve essentially only two steps to a cycle. Convergence of this method for (1) is said to have been proved.

All details of programming this method of solution on punched-card machines in two ways are given. The first uses a tabulator of type D-11 described by Akušskiĭ [Uspehi Matem. Nauk (N.S.) **2**, no. 2(18), 79–184 (1947); these Rev. **10**, 155; see also the review of a paper by Akušskiĭ, Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestiya Akad. Nauk SSSR] **1947**, 475–495; these Rev. **9**, 105] and a reproducer. The sums of products are formed with the tabulator by putting one factor in binary form as previously explained by Akušskiĭ [Doklady Akad. Nauk SSSR (N.S.) **60**, 5–8 (1948); these Rev. **9**, 622]. The second way uses a multiplier (RL2) and sorter. The methods are briefly contrasted with regard to time per cycle, number of cards needed, etc.; each has certain advantages.

R. Church (Annapolis, Md.).

Meerovič, Ê. A. An electrical apparatus for the solution of systems of linear algebraic equations. Êlektričestvo **1947**, no. 4, 65–67 (1947). (Russian)

There are two sets of busses, one vertical and one horizontal. Each vertical bus is connected to each horizontal bus by means of a resistor. The conductances of these n^2 resistors represent the coefficients of the system. Each horizontal bus is connected through an ammeter to ground. The potential of the ith vertical bus with respect to ground (representing the ith unknown) is adjusted so that the reading on the ith ammeter equals the constant term of the ith equation. These adjustments are performed in succession in accordance with the usual Gauss-Seidel procedure. The author compares the circuit with the rather similar one of M. Reck [Arch. Elektrotechnik **32**, 190–197 (1938)] and claims several advantages. No details are given regarding construction, control of inputs or measurement of the unknowns. Convergence of the iterative process is not discussed. Various sources of error are considered but the results of applications are not reported.

R. Church (Annapolis, Md.).

Stelson, Hugh E. The accuracy of linear interpolation in tables of the mathematics of finance. Math. Tables and Other Aids to Computation **3**, 408–412 (1949).

This is a study of the maximum error due to linear interpolation (a) for unknown time, (b) for unknown rate of interest in the $(1+i)^n$, $s_{\overline{n}|}$, v^n, $a_{\overline{n}|}$ tables.

E. Lukacs (China Lake, Calif.).

Rubbert, F. K. Quadratische interpolation bei grossen Differenzen. Z. Angew. Math. Mech. **29**, 54 (1949).

The usual interpolation formulae fail when the differences are large. Using Thiele's reciprocal differences the author finds the two interpolation formulae (where the argument of the table is supposed to have the difference 1): $f(x_0+h)$ is between $y_0+h(y_1-y_0)/[1+(1-h)\kappa]$ and $y_{-1}+(1+h)(y_0-y_{-1})/(1-h\lambda)$, where

$$\kappa=(y_2-2y_1+y_0)/(y_2-y_0),\quad \lambda=(y_1-2y_0+y_{-1})/(y_1-y_{-1}).$$

The results are very satisfactory.

E. Bodewig (The Hague).

Vernotte, Pierre. Sommation, en termes finis, des séries de Fourier les plus générales, en tout point hors de l'intervalle où elles ont été formées. C. R. Acad. Sci. Paris **227**, 966–968 (1948).

Remarks on the extrapolation of a (nonperiodic) trigonometric series. *R. P. Boas, Jr.* (Providence, R. I.).

Vernotte, Pierre. Sommation pratique de séries lentement convergentes. Application aux séries de Fourier, telles qu'on les rencontre dans les problèmes de thermocinétique. C. R. Acad. Sci. Paris **227**, 1015–1016 (1948).

The author indicates an approximate method for summing (nonperiodic) trigonometric series arising in heat-conduction problems. Only experimental justification is cited.

R. P. Boas, Jr. (Providence, R. I.).

Goodwin, E. T. The evaluation of integrals of the form $\int_{-\infty}^{\infty} f(x)e^{-x^2}dx$. Proc. Cambridge Philos. Soc. **45**, 241–245 (1949).

The author makes a study of the error involved in the evaluation of $\int_{-\infty}^{\infty} f(x)e^{-x^2}dx$ by summing the rapidly convergent series $h\sum_{n=-\infty}^{\infty} f(nh)e^{-n^2h^2}$ at a fairly wide interval h. He shows by extending a method due to Turing [Proc. London Math. Soc. (2) **48**, 180–197 (1943); these Rev. **5**, 173] that so long as $f(x)$ does not tend rapidly to infinity with increasing x the error can be estimated from $E(h)=2\pi^{\frac{1}{2}}e^{-\pi^2/h^2}f(i\pi/h)$. Several examples are worked out for which $f(x)$ is a function for which the integration can be carried out exactly. It is found that the error estimated from the formula using values of h of the order of 1 exceeds the actual error by a small amount. Applications to functions $f(x)$ for which the integration cannot be carried out exactly are also given. *S. Levy* (Washington, D. C.).

Mikeladze, Š. E. Numerical integration. Uspehi Matem. Nauk (N.S.) **3**, no. 6(28), 3–88 (1948). (Russian)

This is an exposition of numerical differentiation and integration, with some applications to the solution of ordinary differential equations. Formulas for differentiation with and without differences are derived in the first chapter. Some sections are devoted to related topics, so that the derivations are essentially complete. Then a longer chapter is used for quadrature formulas, including formulas of Gauss, Čebyšev, and Markov, and a discussion of S. N. Bernstein's work on quadrature formulas. Sections are also devoted to integrals of the type $\int_a^b F(x)f(x)dx$ and to Stieltjes integrals.

Quadrature formulas involving fixed ordinates are found from an interpolation formula stated to have been previously derived by the author in a book in Georgian. To get this formula, the Taylor series for $y^{(\lambda+1)}(x)$, with powers of $x-A$, is integrated over the interval (A, x). When the integrand in the remainder term is replaced by Lagrange's interpolation formula, Mikeladze's general formula for $y^{(\lambda)}(x)$ is obtained. In the third chapter, the author uses special cases of this formula to solve ordinary differential equations by a method analogous to that of Adams. Since different derivatives, differences and ordinates appear in the same quadrature formula, his method generalizes that of C. Störmer [see Bennett, Milne, and Bateman, Bull. Nat. Res. Council, no. **92** (1933), p. 82] and one attributed to A. N. Krylov. Finally, two sections are devoted to the author's iteration method for numerical integration of differential equations.

R. E. Gaskell (Ames, Iowa).

Sard, Arthur. Best approximate integration formulas; best approximation formulas. Amer. J. Math. 71, 80–91 (1949).

The author studies approximations of $I=\int_0^m x(t)dt$ by $A=c_0x_0+c_1x_1+\cdots+c_mx_m$, where $x_i=x(i)$, under the conditions: (a) $A=I$, when x is a polynomial of degree n in t (then generally $I-A=\int_0^m x^{(n+1)}(t)k(t)dt$ when $x^{(n)}$ is absolutely continuous); (b) $J=\int_0^m k^2(t)dt$ is a minimum for all functions satisfying (a). The author proves that there is a unique A (if any) for every m and n. He gives a table of A for $m\leqq 6$, $n\leqq 3$. The problem is generalized in the last section of the paper. *E. Bodewig* (The Hague).

Salzer, Herbert E. Coefficients for repeated integration with central differences. J. Math. Physics 28, 54–61 (1949).

If Everett's interpolation formula for $f(x_0+ph)$ is integrated k times the k-fold integral of $f(x)$ between x_0 and x_1 will be

$$h^k[A_0^{(k)}f_0+B_0^{(k)}f_1+\sum_{s=1}^{m}(A_{2s}^{(k)}\delta_0^{2s}+B_{2s}^{(k)}\delta_1^{2s})]+R_{2m}.$$

The tables give (a) $A_{2s}^{(2)}$, $B_{2s}^{(2)}$ for $2s=0(2)20$ (exact values) and for $2s=22(2)48$ (to 16 decimal places); (b) $A_{2s}^{(k)}$, $B_{2s}^{(k)}$ for $k=3, 4, 5, 6$ and $2s=0(2)20$ to 8 figures. Various relations of the A_i and B_i and other quantities are given.

E. Bodewig (The Hague).

Heinrich, H. Genauigkeitsvergleich für die Halbschnittverfahren der graphischen Integration. Z. Angew. Math. Mech. 29, 51–52 (1949).

The author is concerned with the comparative accuracy of three graphical methods of integrating the first order equation $dy/dx=f(x, y)$. The three procedures are (a) the "chord construction," (b) the tangential construction, (c) the tangential construction followed by iteration. Whilst in the case of $f_y(f_x+ff_y)=0$ (e.g., plain quadrature of $f(x)$) (a) is twice as accurate as either (b) or (c), the comparative accuracy will in general depend on the "index"

$$K=(f_{xx}+2ff_{xy}+f^2f_{yy})/f_y(f_x+ff_y)$$

zoning the (x, y)-plane for any given $f(x, y)$ into regions in which (a), (b) or (c) is optimum. The example $dy/dx=(1+x^2)^{-1}$ is discussed. *H. O. Hartley.*

Rabinovitch, Féodora. Sur une nouvelle méthode d'intégration approchée des équations différentielles du second ordre. Ann. Radioélec. 1, 134–151 (1945).

The author develops a new method of numerical integration of second order differential equations of the type $d^2x/dt^2=f(x)\phi(t)$ with particular reference to the case $\phi(t)=\sin t$ which is of importance in the movement of a particle in an oscillating field. The method is a step by step integration at equal t-intervals using finite differences of $f(x(t))$ (rather than those of the product $f\phi$) and is based on the approximate representation of $f(x(t))$ by its polynomial interpolation formula. This leads to a "backward" integration formula of the type

$$\Delta^2x_{n-1}=A_nf_n+B_n\Delta f_{n-1}+C_n\Delta^2f_{n-2}+D_n\Delta^3f_{n-3},$$

where the Δf_n are advancing differences of f and the $A_n, \cdots, D_n$ are coefficients depending on $\phi(t)$ which are tabulated for $\phi(t)=\sin t$. The method is illustrated for this case with several examples, compared with Störmer's method and found superior. Formulae for gauges of accuracy are derived. For the method to be usable, integrals of the type

$$\int_{t_0}^{t_0+h} t^k\phi(t)dt$$

must be evaluated and tabulated in advance.

H. O. Hartley (Princeton, N. J.).

Mikeladze, Š. E. The numerical integration of differential equations by means of summation formulas. Doklady Akad. Nauk SSSR (N.S.) 65, 125–128 (1949). (Russian)

Using the notation $y_r^{(k)}$ for the value $y^{(k)}(a+rh)$ of a function $y(x)$ which is the solution of the differential equation $y^{(n)}=f(x, y, y', \cdots, y^{(n-2)})$, with the initial conditions $y(a)=y_0, y'(a)=y_0^{(1)}, \cdots, y^{(n-1)}(a)=y_0^{(n-1)}$, and defining the sum

$$S^{p+1}y_{r+1}^{(n)}=\sum_{\nu=0}^{r-\mu}\binom{r-\nu}{\mu}y_\nu^{(n)},$$

we have the identity

$$S^m y_{i-1}^{(n)}+S^{m-1}y_{i-1}^{(n)}\equiv S^m y_i^{(n)}. \tag{1}$$

From Taylor's formula, approximating the remainder in the integral form by the trapezoid rule, we find

$$y_r^{(k)}=\sum_{\lambda=0}^{n-k-1}\frac{(rh)^\lambda}{\lambda!}y_0^{(k+\lambda)}-\tfrac{1}{2}\frac{r^{n-k-1}h^{n-k}}{(n-k-1)!}+\frac{h^{n-k}}{(n-k-1)!}M_{n-k-1,r}+R_{r,k}, \tag{2}$$

where

$$M_{n-k-1,r}=\sum_{\nu=0}^{r-1}(r-\nu)^{n-k-1}y_\nu^{(n)}=\sum_{\mu=1}^{n-k-1}c_{n,n-k-1}S^{\mu+1}y_{r+1}^{(n)},$$

with universal constants $c_{n,p}$ defined by

$$(r-\nu)^p=\sum_{\mu=1}^{p}\binom{r-\nu}{\mu}c_{n,p}.$$

Neglecting $R_{r,k}$, the values $y_1^{(k)}, y_2^{(k)}, \cdots$ can be obtained successively from $y_0^{(k)}$ [see (2)] and the given equation and from (1) it follows that the S^p can be tabulated in a difference scheme. The method can be used to calculate eigenvalues; e.g.,

$$y''=-\lambda f(x)y(x), \quad y(0)=0, \quad y'(0)=1, \quad y(1)=0$$

gives

$$y_r=rh-\lambda h^2\sum_{\nu=1}^{r-1}(r-\nu)f_\nu y_\nu, \; y_1=h, \; y_2=2h-\lambda h^2 f_1, \cdots.$$

Taking $h=\frac{1}{2}, \frac{1}{3}, \cdots$, approximations for λ can be found from the algebraic equations $y_2=0, y_3=0, \cdots$. With $f(x)=x$ the smallest values of λ for $h=\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$ are 16, 17.1, 18.25, the exact value being 18.96. *E. M. Bruins* (Amsterdam).

Pentkovskiĭ, M. V. A nonprojective transformation of nomograms of equations of the third nomographic order. Doklady Akad. Nauk SSSR (N.S.) 62, 589–590 (1948). (Russian)

A summary equivalent to the résumé of R. Soreau [Nomographie ou Traité des Abaques, Chiron, Paris, 1921, v. 2, p. 98] is given for the geometrical features of the eight representations for Soreau's three canonical forms [loc. cit., v. 1, p. 172] of the equation of the third nomographic order. A generalization of the nonprojective transformation mentioned by Soreau [loc. cit., v. 2, p. 94] which does not alter the carriers of the scales but changes the labels except for those of the critical points is briefly discussed and some practical uses mentioned. It is not made clear whether any of the material of the paper is thought to be original.

R. Church (Annapolis, Md.).

Vil'ner, I. A. A pencil of conics for the representation of the elliptic integral of the first kind by means of an alignment nomogram. Uspehi Matem. Nauk (N.S.) **2**, no. 6(22), 227–237 (1947). (Russian)

The present paper is concerned with the description of and effective use of an alignment chart which is to be included in a forthcoming album of nomograms concerned with elliptic functions. The disjunction of the variables and the equations for the determination of the scales are to be given later. The nomogram, the existence of which was previously shown [C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 187–190 (1946); these Rev. **8**, 494], permits the immediate determination by means of one alignment of any two of the quantities p_{k_1}, q_{k_1}, p_{k_2}, q_{k_2} from the other two when k_1 and k_2, $0 \leqq k_j \leqq 1$, are given, where

$$p_{k_j}+iq_{k_j}=\int_0^{w_0}(1-k_j^2\sin^2\xi)^{-\frac{1}{2}}d\xi, \qquad w_0=p_0+iq_0,$$

for $j=1, 2$, and in terms of the complete elliptic integral $0 \leqq p_{k_j} \leqq K(k_j)$, $0 \leqq q_{k_j} \leqq K'(k_j)$. The curves for the various values of k_j are arcs of a pencil of hyperbolas, three of the four points common to the conics of the pencil appearing in the diagram. The calibrations for p_{k_j} (shown by means of a family of transcendental curves) are on one portion of these hyperbolic arcs and those for q_{k_j} on the other. When the degenerate conics of the pencil corresponding to $k_j=0$ and $k_j=1$ have been properly recalibrated, the nomogram presented earlier by the author is obtained [Akad. Nauk SSSR. J. Appl. Math. Mech. N.S. **4**, no. 1, 145–152 (1940); these Rev. **9**, 534]. The paper closes with brief allusions to technical details of producing such a family of curves and of transforming nomograms by photographic means and by mapping on a sphere. *R. Church* (Annapolis, Md.).

Vil'ner, I. A. Nomography of analytic functions. Doklady Akad. Nauk SSSR (N.S.) **63**, 99–102 (1948). (Russian)

Continuing his study [especially Akad. Nauk SSSR. J. Appl. Math. Mech. N.S. **4**, no. 2, 105–116 (1940); C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 187–190 (1946); these Rev. **9**, 534; **8**, 494] of the conditions under which an analytic function $F(z, w)=0$ can be represented by an alignment chart with two scales for z and two scales for w, the author introduces parameters K_{j_s} ($j=1, 2, 3, 4$) and J_{j_s} ($j=1, 2, 3$) depending on the first four derivatives of w with respect to z. The representation in question is possible if and only if all of the K_{j_s} are real and constant. In that case all scales are curved if and only if K_{1_s} and K_{4_s} are both different from zero while if K_{1_s} (or K_{4_s}) only is zero w (or z) only has straight scales and finally if both K_{1_s} and K_{4_s} are zero all scales are straight. Canonical forms are given for $F(z, w)=0$ if it can be nomographed as indicated. They take the form of the integral of the reciprocal of the square root of a linear function of the Weierstrass elliptic function. The invariants g_2 and g_3 are given in terms of the parameters K_{j_s}. Details of applying these results are to be given elsewhere. *R. Church* (Annapolis, Md.).

Wolf, H. Betrachtungen zur astronomisch-geodätischen Netzausgleichung unter besonderer Berücksichtigung der Laplaceschen Gleichung. Astr. Nachr. **276**, 209–220 (1949).

ASTRONOMY

Široký, J. On the Wilkens method concerning the determination of the orbit of a planet or a comet. Publ. Fac. Sci. Univ. Masaryk, no. **290**, 14 pp. (1947).

This article contains a modification of a method for computing a planetary or cometary orbit from three observations, developed by A. Wilkens [Astr. Nachr. **210**, 81–112 (1920)]. The principal change is in the fundamental equation from which the heliocentric distance for the middle observation is determined. This equation is here given in a more complete form than the corresponding equation by Wilkens. *D. Brouwer* (New Haven, Conn.).

Samoilova-Jachontova, N. Quelques corrections aux tables de Bohlin pour le calcul des perturbations absolues des petites planètes. Akad. Nauk SSSR. Bull. Inst. Teoret. Astr. **4**, 43–56 (1947). (Russian. French summary)

Le mémoire de Bohlin "Sur le développement des perturbations planétaires" [Stockholm, 1902] contient une erreur dans l'intégration des équations. Dans les termes du second ordre par rapport à la petite quantité w Bohlin met

(1) $$w+(1-w)W=(1-w)W_1+(1-w)V$$

en désignant par $(1-w)V$ l'ensemble des termes dépendant explicitement de l'anomalie excentrique ϵ, et par $(1-w)W_1$ la somme de tous les autres termes. W et V sont de l'ordre de la masse troublante et W_1 contient, comme il est évident par la formule (1) un terme indépendant de la masse troublante m': $w/(1-w)$. Néanmoins Bohlin adopte dans tous ses calculs que le terme indépendant de m' est égal a w, omettant les termes en w^2. L'ouvrage présent contient les formules et les tables corrigées pour le calcul des perturbations des petites planètes avec le mouvement diurne près de 900″. *Author's summary.*

Iacob, Caius. On a necessary condition for the transformation into satellites of the planet of bodies projected from a planet. Gaz. Mat., Bucureşti **54**, 192–201 (1949). (Romanian)

Batyrev, A. A. The shape of the trajectories in the problem of two bodies with variable masses. Akad. Nauk SSSR. Astr. Zhurnal **26**, 56–59 (1949). (Russian)

The author considers the case in which the sum of the masses of the two bodies varies according to the law $M=M_0/(1+\alpha t)$, with $\alpha>0$. Introducing the new variables

$$\xi=\frac{x}{1+\alpha t}, \quad \eta=\frac{y}{1+\alpha t}, \quad \tau=\frac{1}{\alpha(1+\alpha t)},$$

the equations of motion of the auxiliary point (ξ, η, τ) are identical in form with the equations of motion of the point (x, y, t) when the mass of the system is constant; i.e., the auxiliary point describes a conic. The author examines the four cases in which the auxiliary point describes: (1) an ellipse with low eccentricity, (2) an ellipse with large eccentricity, (3) a parabola, (4) a hyperbola. The positions of the auxiliary point are converted to (x, y) coordinates and the resulting orbits are shown graphically. The trajectories are spiral-like; those corresponding to the parabola and the hyperbola have asymptotes. *L. Jacchia.*

Musen, Peter. Zur Reduktion des Dreikörperproblems. Z. Naturforschung **3a**, 360–363 (1948).

The theory of transformations of Pfaffians is applied to obtain a brief derivation of the well-known reduction of the problem of three bodies to a system of differential equations of the eighth order. *M. H. Martin* (College Park, Md.).

Lemaître, G., et Vander Borght, R. Modèles de nébuleuses à vitesses radiales. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 955–965 (1948).

The paper is concerned with the study of equilibrium configurations for stellar systems with radial symmetry and in which only radial and no transverse velocities are permitted. Two models are studied. In the first model, the stars are supposed to oscillate through the center with one definite positive or negative velocity at each given distance from the center. The nature of the radial density distribution is determined by the condition that a stationary condition is established under the influence of the forces exerted by the stars of the system together on each individual member. Relativity effects are ignored.

In the second model there is a continuous distribution of radial speeds at each point with the maximum velocity assigned to the stars which oscillate from one point at the boundary of the cluster through the center to the opposite point on the boundary, but with smaller velocities and correspondingly more limited ranges of oscillation permitted for other stars of the cluster.

These models give evidence for irregularities at the center, which have some resemblance to the observed ring effects in some galaxies with the exception of the center and the outer boundary. The derived density distributions agree well with the observations, which was not the case with a model previously studied by Lemaître. *B. J. Bok.*

Pignedoli, Antonio. Sul moto di rotazione di una massa liquida omogenea nel campo gravitazionale di più centri lontani. Atti Sem. Mat. Fis. Univ. Modena **1**, 59–77 (1947).

The author studies the motion of rotation of a homogeneous liquid mass, bounded by the surface of an ellipsoid, subject to gravitational forces due to n distant masses, situated in the plane of motion of the baricentre O of the liquid mass, and rotating slowly and uniformly about O with different angular velocities. The variation of the angular velocity of a particle of the liquid mass as function of the distances to the centres of attraction is given with an application to the motion of the earth, considered as homogeneous liquid, subject to two attractive centres represented by the Sun and the Moon. It is shown that the terrestrial angular acceleration varies periodically according to a quasi-sinusoidal law. The numerical values for limits of the amplitude of oscillation are also given. [Cf. Agostinelli, Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **77**, 130–153 (1942); these Rev. **7**, 494.] *E. Leimanis* (Vancouver, B. C.).

Underhill, Anne B. The Schuster problem for an extended atmosphere. Astrophys. J. **107**, 247–264 (1948).

The author considers the Schuster problem (the formation of absorption lines) in a stellar atmosphere, taking the sphericity of the atmosphere into account. The problem is reduced to the equation of transfer

$$\cos\theta\frac{\partial I_\tau}{\partial\tau}-\frac{\sin\theta}{r}\frac{\partial I_\tau}{\partial\theta}=-l_{,\rho}I_\tau+\tfrac{1}{2}l_{,\rho}\int_0^\pi I_\tau(r,\theta')\sin\theta'd\theta'.$$

An approximate solution for an atmosphere of infinite depth has been obtained by Chandrasekhar [same J. **101**, 95–107 (1945); these Rev. **6**, 244] in the case $l_{,\rho}\propto r^{-n}$ $(n>1)$. The author obtains an improved approximation by introducing a correction term in Chandrasekhar's solution, and applying the appropriate boundary condition. The general conclusion is that when the optical depth of the atmosphere is large enough for its sphericity to have an appreciable influence, the cores of the absorption lines are shallower and the wings deeper than for a plane-parallel atmosphere.

F. Smithies (Cambridge, England).

RELATIVITY

***Milne, E. A. Kinematic Relativity. A Sequel to Relativity, Gravitation and World Structure.** Oxford, at the Clarendon Press, 1948. vii+238 pp. $6.50.

This book is a self-contained account of Milne's theory. "Kinematic relativity" is essentially a system of physical laws which is invariant under the group of homogeneous Lorentz transformations. The time axes of the equivalent Lorentz frames are identified with the nuclei of galaxies, the common space-time origin O of the frames with the event of creation of the universe. Milne's theory differs considerably from even special relativity. The coordinates of the event of creation O enter explicitly into the dynamical equations. Thus, for example, the author is able to construct an invariant which he identifies with the energy of a particle. The book is divided into four roughly equal parts dealing respectively with kinematics, dynamics, gravitation, and electrodynamics. The author uses three-dimensional vector notation almost exclusively. In the reviewer's opinion, the more suitable and compact four-dimensional tensor notation would often have improved the presentation.

A. Schild (Pittsburgh, Pa.).

Causse, Maurice. Les notions de masse et d'énergie et le problème des deux corps en relativité cinématique. C. R. Acad. Sci. Paris **228**, 542–544 (1949).

The two-body problem in Milne's kinematic relativity is discussed. *A. Schild* (Pittsburgh, Pa.).

Qvist, Bertil. On a spherically symmetric mass concentration in an expanding universe. Soc. Sci. Fenn. Comment. Phys.-Math. **13**, no. 11, 12 pp. (1948).

An isotropic line element of the form

$$ds^2=e^{\nu(r,t)}dt^2-e^{\mu(r,t)}(dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\varphi^2)$$

is considered and the gravitational equations for empty space are solved to determine μ, ν. The final form of solution is left as an integral which could have been evaluated by means of elliptic functions. [This solution was originally determined by H. P. Robertson; see Trans. Amer. Math. Soc. **29**, 481–496 (1927), p. 492.] *M. Wyman.*

Kustaanheimo, Paul. Some remarks concerning the connexion between two spherically symmetric relativistic metrics. Soc. Sci. Fenn. Comment. Phys.-Math. **13**, no. 12, 8 pp. (1948).

The author generalizes some previous work by Qvist [see the preceding review]. By using the fact that the covariant derivative $T^4_{j;4}=0$ for the energy momentum tensor the author could have derived equation (16) of this paper. In this way it would have been seen that another relation connecting the pressures p_1, p_2 and the density ρ also exists, namely

$$p_1'+\left(\frac{2}{r}+\mu'+\frac{\nu'}{2}\right)p_1-\left(\frac{2}{r}+\mu'\right)p_2+\tfrac{1}{2}\nu'\rho=0.$$

From this relation the author's assumption 5(a) that $p_1=p_2=0$ would immediately imply that $\rho=0$ or $\nu'=0$. Both of these cases would lead to known solutions and hence the first three solutions listed on page 5 would not lead to new solutions. *M. Wyman* (Edmonton, Alta.).

Kustaanheimo, Paul, and Qvist, Bertil. A note on some general solutions of the Einstein field equations in a spherically symmetric world. Soc. Sci. Fenn. Comment. Phys.-Math. **13**, no. 16, 12 pp. (1948).

The authors consider an isotropic nonstatic line element of the form

$$(1)\qquad ds^2=e^{\nu(r,t)}dt^2-e^{\mu(r,t)}(dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\varphi^2)$$

and proceed to show that the gravitational field equations for a perfect fluid at rest can be reduced to an equation of the form (2) $d^2y/dx^2=f(x)y^2$. By choosing $f=(ax^2+bx+c)^{-6/7}$, the authors proceed to find seven solutions of the field equations. They make the conjecture that by their method they have probably determined all such nonstatic spherically symmetric solutions which can be expressed in terms of elementary functions. This conjecture is not correct. For example $f=x^{-20/7}$ or $f=x^{-15/7}$ both lead to solutions, not listed by the authors, which can be expressed as elementary functions.

It should be noted that the authors state that every spherically symmetric metric can be written in the so-called isotropic form. This assertion is incorrect as it is based on a proof given by Tolman [Relativity, Thermodynamics and Cosmology, Oxford, 1934, p. 240] which is wrong. The right hand side of formula (94.7) as given by Tolman is not a perfect differential when λ is a function of t.

M. Wyman (Edmonton, Alta.).

Narlikar, V. V., and Prasad, Ayodhya. Canonical coordinates in general relativity. Bull. Calcutta Math. Soc. **40**, 123–128 (1948).

The authors solve the equations defining a canonical coordinate system and determine the values of the metric tensor up to and including second order terms. Using these values it is shown that the velocity of light is nonisotropic and that it will exceed its special relativity value in some direction. [In considering the equations of geodesics in a canonical coordinate system the reviewer would like to point out that, under the condition that the particle is initially at rest at the origin, the equations have a unique solution, namely, $x=y=z=0$. Hence the only conclusion that can be made is that, in such a coordinate system, a test particle at rest at the origin, initially, will remain at rest at the origin.]

M. Wyman (Edmonton, Alta.).

Clark, G. L. The gravitational field of a rotating nearly spherical body. Philos. Mag. (7) **39**, 747–778 (1948).

The author considers the linearized equations of general relativity and finds solutions for the gravitational potentials which can be interpreted as representing the gravitational field due to a rotating nearly spherical body. Expressions are obtained for both the internal and external fields. It is assumed that the angular velocities about the principal axes are small enough so that cubes and higher powers can be neglected. A short discussion on the values of the energy tensor is also included. *M. Wyman* (Edmonton, Alta.).

Van Bergen, F. Transformation de la forme quadratique fondamentale de l'univers de De Sitter. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 975–977 (1948).

By embedding the De Sitter universe in a five-dimensional Euclidean space it is shown that this universe is a Cayleyan space. *M. Wyman* (Edmonton, Alta.).

Ludwig, Günther. Fortschritte der projektiven Relativitätstheorie. Arch. Math. **1**, 212–218 (1948).

This paper is a short outline of a forthcoming book. Following a study of projective differential geometry, variational principles are considered which yield physical field equations. A cosmological model is obtained which satisfies the field equations, and the connection with the recent work of P. Jordan is discussed. *A. Schild* (Pittsburgh, Pa.).

Fihtengol'c, I. G. On integrals of the motion of the center of inertia of a system of finite masses in the general theory of relativity. Doklady Akad. Nauk SSSR (N.S.) **64**, 325–327 (1949). (Russian)

This is a generalization of the work of Fock [C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 25–27 (1941); these Rev. **3**, 212] dealing with the motion of the center of mass of two finite masses. Starting with the approximate equations of motion for a system of masses described by means of harmonic coordinates which is said to have been obtained by N. M. Petrova [Dissertation, Leningrad State University, 1940] on the basis of earlier work of Fock [Acad. Sci. U.S.S.R. J. Phys. **1**, 81–116 (1939); these Rev. **1**, 183], the author derives the equations of motion for the center of mass of the system. He finds that the acceleration of the center of mass in the general case is different from zero. However, on modifying the definition of the center of mass so as to include terms involving the kinetic and potential energy of the masses, he obtains the result that the acceleration of this generalized center of mass vanishes, to within the accuracy of the approximation employed.

N. Rosen (Chapel Hill, N. C.).

Schild, Alfred. Discrete space-time and integral Lorentz transformations. Canadian J. Math. **1**, 29–47 (1949).

This paper is chiefly concerned with those Lorentz transformations (with $c=1$) that map the "cubic lattice" onto itself, the "cubic lattice" being the set of points with real integer coordinates t, x, y, z in Minkowski space-time. These transformations are linear with integer coefficients, and the author makes use of spin tensors with Gaussian integer components in their study. A relativity theory based on a discrete set of space-time events may prove to be significant physically. With this possibility in mind, the author proves the following theorems. (1) The spatial projections of the (forward) null lines joining a fixed point to other lattice points are dense. (2) Any two such null lines can be transformed into each other by integral Lorentz transformations. (3) The spatial projections of the (forward) time-like lines joining a fixed point to other lattice points are dense. From (3) it follows that the directions in which material particles can move form a dense system (a particle can move in "approximately" any direction). The scalar speeds, however, take only discrete values, the smallest nonzero value being $\frac{1}{2}\sqrt{3}$, not far short of the speed of light. In an appendix the author refers to some [unpublished] work of H. S. M. Coxeter, in which very much the same kind of problem is treated by a different and perhaps simpler method.

A. G. Walker (Sheffield).

Papapetrou, A. Static spherically symmetric solutions in the unitary field theory. Proc. Roy. Irish Acad. Sect. A. **52**, 69–86 (1948).

The general form of a spherically symmetric tensor field g_{ik} is found to have the form

$$g_{ik}=\begin{bmatrix} -\alpha & 0 & 0 & w \\ 0 & -\beta & r^2 v \sin\theta & 0 \\ 0 & -r^2 v \sin\theta & -\beta \sin^2\theta & 0 \\ -w & 0 & 0 & \gamma \end{bmatrix},$$

where α, β, γ, w are four arbitrary functions of r. The unitary field equations are obtained for the two cases $v=0$, $w\neq 0$ and $w=0$, $v\neq 0$. For the first case the general solution of the equations is obtained and in the second a special solution is derived. In each case the author compares the solutions obtained with the corresponding solution obtained by the general theory of relativity and finds significant differences in these solutions. The paper concludes by deriving the equations determining the second approximation for weak fields and also a short discussion on the physical interpretation of the mathematical quantities entering into the field equations. *M. Wyman* (Edmonton, Alta.).

Votruba, Václav. On the Lorentz covariance of the field equations. Rozpravy II. Třídy České Akad. **56**, no. 11, 23 pp. (1946). (Czech)

The author's aim is to find all admissible nonequivalent transformation rules for field functions which leave a given set of field equations covariant under Lorentz transformation. As all transformations of nonlinear equations can be reduced to transformations of linearised ones, the author studies linear differential equations only. By consistent use of spinor calculus, he finds the method of which the practical application is the following. The operators $\partial/\partial x^i$ are replaced by the components of the corresponding mixed spinor of second rank and, by taking linear combinations, the original set of equations are transformed to an equivalent set, in which the new field functions can be identified as components of certain spinors. The dependence of the original field functions on the spinor components then follows by the solution of linear algebraic equations.

Using this method the author investigates the Frenkel-Madelung equations for particles with zero rest-mass [generalised "Maxwell-like" equations; Frenkel, Wave Mechanics, Advanced General Theory, Oxford, 1934, p. 264]; he finds for this set six nonequivalent transformation rules. He further demonstrates that the considerations of the authors on the transformational properties of the complete set of Frenkel-Madelung equations for the electron [Frenkel, loc. cit., p. 266; Madelung, Z. Physik **54**, 303–306 (1929); **57**, 573–574 (1929)] are confused. *M. Brdička* (Prague).

Hardtwig, Erwin. Über Differentialgleichungen mit Invarianz gegenüber der Lorentzgruppe. Ann. Physik (6) **2**, 273–285 (1948).

Lorentz invariant differential equations are obtained as limiting cases of equations which are invariant under the projective group of the nondegenerate quadric

$$\pm x(x_1^2+x_2^2+x_3^2-x_0^2)+1=0.$$

Maxwell's equations are obtained in this manner; also a wave equation which shows some similarity to Dirac's equation. *A. Schild* (Pittsburgh, Pa.).

Rumer, Yu. B. Action as coordinate of a space. I. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **19**, 86–94 (1949). (Russian)

The author considers the motion of a single particle in a gravitational and electromagnetic field. For this problem there is a classical Hamilton-Jacobi equation

$$(1)\qquad g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k}+2\frac{e}{c}A^i\frac{\partial S}{\partial x^i}+\left(m^2c^2+\frac{e^2}{c^2}A_iA^i\right)=0.$$

A more symmetrical form of (1) is obtained by considering the action-function $(S/mc)=x_5$ as a fifth coordinate for the particle. In 5-dimensional space, possible motions are related to solutions of a problem in optics. In classical theory, the problem is in geometrical optics, the eikonal function Σ satisfying an equation (2) $G^{\mu\nu}(\partial\Sigma/\partial x^\mu)(\partial\Sigma/\partial x^\nu)=0$. In quantum theory, the analogy is with wave optics; a wave-equation for the particle is constructed which leads back in the classical limit to (2), and reduces in absence of gravitational fields to the Dirac equation. The wave-function has a simple exponential dependence on x_5, and the momentum conjugate to x_5 is a constant of the motion equal to the charge of the particle. The exposition of these ideas is highly condensed, and their relation to other 5-dimensional generalizations of relativity theory is not discussed.

F. J. Dyson (London).

Rumer, Yu. B. Action as coordinate of a space. II. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **19**, 207–214 (1949). (Russian)

The author's 5-dimensional theory [see the preceding review] of the motion of a particle in a combined gravitational and electromagnetic field is here developed as a classical unified field theory. The field equations are assumed to be the Einstein equations modified by the addition of the fifth coordinate. It is shown that these equations, under special conditions, reduce to the standard form of coupled Einstein and Maxwell equations. In the general case, there appears, besides the Einstein and Maxwell fields, a scalar field. The fields around a stationary point singularity are not given precisely by the Schwarzschild solution and the Coulomb law; in fact, the electric potential remains finite at the singularity.

It is supposed that the "fundamentons," the original particles whose motions were used to define the fifth coordinate, possess a variety of states of differing charge and spin. In its zero-charge states, a fundamenton may appear either with spin 2 (graviton), spin 1 (photon) or spin 0 (scalar particle). However, the relation between the fundamenton and the Maxwell field (for example) is certainly very different from the relation between photon and Maxwell field in orthodox quantum electrodynamics, and it is not made clear how these two types of particle are to be equated.

F. J. Dyson (London).

Scherrer, W. Gravitationstheorie und Elektrodynamik. Helvetica Phys. Acta **22**, 89–100 (1949).

A relativistic nonlinear electromagnetic theory is derived from the Lagrangian

$$\tfrac{1}{2}(-g)^{\frac{1}{2}}(F_{\lambda\alpha}F^{\lambda\alpha}+\epsilon F_{\lambda\alpha}F^{\alpha\mu}F_{\mu\beta}F^{\beta\lambda}),$$

where ϵ is a small constant. A static, spherically symmetric solution is obtained which represents an electron of finite electromagnetic mass. *A. Schild* (Pittsburgh, Pa.).

Gião, Antonio. Sur les rapports entre gravitation et électromagnétisme déduits des équations de Codazzi. Application au champ électromagnétique général des astres. C. R. Acad. Sci. Paris **228**, 742–744 (1949).

Gião, Antonio. La constante cosmologique gravifique et les équations de Gauss d'une hypersurface. C. R. Acad. Sci. Paris **228**, 812–813 (1949).

The author assumes without justification that the underlying space of general relativity is a hypersurface in a flat five-dimensional space. In the first paper he uses the Codazzi equations to obtain relations between the metric tensor (gravitation) and some components of the second fundamental form (these are supposed to represent the four-vector potential of electromagnetic theory). A relation between magnetic moment and angular momentum is given. It is a consequence that every static gravitational field has associated with it an electrostatic field.

In the second paper the Gauss equations are used to obtain a relation between the cosmological constant of general relativity, the components of the two forms of the hypersurface, and the stress energy tensor.

A. H. Taub (Urbana, Ill.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Maréchal, André. Sur les aberrations géométriques des systèmes optiques faiblement décentrés. C. R. Acad. Sci. Paris **228**, 668–670 (1949).

The author considers an optical system consisting of two parts A and B such that the axis of the second system is slightly tilted with respect to the first. The change of the terms of third order with decentration are analyzed and their optical significance is given. The higher order aberrations introduced by decentration lead to specific new aberration types which cannot be interpreted as the aberrations of a well centered system. *M. Herzberger.*

Di Jorio, M. Applications de la théorie générale de la courbure de champ des systèmes optiques aux problèmes de photogrammétrie. Ottica (N.S.) **2**, 13–20 (1948).

The author starts from a system of formulae previously developed by him which compute sagittal and meridional curvature for meridional rays of finite aperture in optical systems with rotational symmetry. The formulae are specialized for third order aberrations and he obtains results which are in agreement with the conclusions from Seidel theory. Under the assumption that the exit pupil is a point, i.e., that all the principal rays go through a point in image space, the author tries to integrate his starting formulae. The reviewer does not understand the transition from formula (1) and (2) to formula (14) of the paper. From his assumption the author derives an equation which connects object and image curvature for finite object and field which, if correct, is a generalization of the well-known Petzval theorem; as such its form is independent of the position of the object. The importance of the results achieved for the correction of optical systems, especially for photogrammetric objectives, is stressed at the end of the article.

M. Herzberger (Rochester, N. Y.).

Crook, A. W. The reflection and transmission of light by any system of parallel isotropic films. J. Opt. Soc. Amer. **38**, 954–964 (1948).

The reflection and transmission coefficients for the particular cases of one, two or three films are calculated as functions of the Fresnel coefficients for each surface and of the optical path for each film. These functions are found to be rather complicated fractions, whose numerators and denominators can be formed by applying a certain rule. The rule is then generalized by induction for an arbitrary number of films. It is then shown that the results obtained by this rule coincide with those derived by elementary summation of multiple beams. Further it is demonstrated that, when the films are nonabsorbing and one surface is perfectly reflecting, the reflection coefficient of the entire system becomes unity, as is required by physical reasons. A previous solution given by Vašiček [J. Opt. Soc. Amer. **37**, 623–634 (1947); these Rev. **9**, 105] does not satisfy this condition and cannot therefore be correct.

G. Toraldo di Francia (Florence).

Durand, Emile. Étude des franges d'interférences au voisinage des caustiques. Ann. Physique (12) **3**, 621–636 (1 plate) (1948).

The first problem discussed in this paper is the rigorous determination of the interference surfaces near the caustic of a concave spherical mirror inclined at an angle of 45° to an incident pencil of parallel rays, and also of the wave-surfaces. The object of this is to enable the author then to calculate the intensity in the system of fringes by two different methods, (a) by geometrical optics and interference, (b) by Weber's modification of Helmholtz's formula for cylindrical waves. *E. T. Copson* (Dundee).

Croze, François, et Durand, Émile. Sur les expressions du principe de Huygens pour les ondes électromagnétiques. C. R. Acad. Sci. Paris **228**, 236–239 (1949).

L. de Broglie [Problèmes de Propagations Guidées des Ondes Electromagnétiques, Gauthier-Villars, Paris, 1941; these Rev. **8**, 183] introduced a modified form of the formulae of Kottler [Ann. Physik (4) **71**, 457–508 (1923)] in which the secondary waves emitted by an element of a wave-surface are due only to superficial distributions of electric and magnetic charge and current. In the present note, it is shown that the de Broglie formulae lead to the known results of Love, Larmor and Bromwich.

E. T. Copson (Dundee).

Croze, François, et Boillet, Pierre. Sur l'expression du principe de Huyghens pour les ondes électromagnétiques. C. R. Acad. Sci. Paris **228**, 305–307 (1949).

In the note reviewed above it was shown that L. de Broglie's modification of the Kottler formulae leads to known results of Love, Larmor and Bromwich. It is now shown that these last results can also be deduced immediately from the classical formulae for the fields generated by electric and magnetic doublets. *E. T. Copson.*

Kuznecov, P. I. The propagation of electromagnetic waves in a multi-conductor system. Uspehi Matem. Nauk (N.S.) **4**, no. 1(29), 209–212 (1949). (Russian)

Summary of the author's dissertation [Moscow, 1948].

Malvano, R. Guida d'onda dielettrica a sezione rettangolare compresa tra due piani metallici paralleli. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 155–161 (1948).

Eliezer, C. Jayaratnam. On the classical theory of particles. Proc. Roy. Soc. London. Ser. A. **194**, 543–555 (1948).

The problem of elimination of nonphysical solutions, e.g., the "self-accelerating" one, of the Lorentz-Dirac equation of classical electrodynamics is discussed. First a nonrelativistic equation of motion for an extended electron is formulated: this consists of an infinite series of terms involving higher derivatives of the velocity: its form is chosen for analytical convenience and it resembles an equation derived by Page [Physical Rev. (2) **11**, 376–400 (1918)]. Applications are given. Next, relativistic equations of motion for a point electron are formulated containing an infinite series of higher derivatives of the velocity. For a motion in which the velocity and all these derivatives are small a limiting form of the equations of motion is obtained. Relativistic equations of motion are then formulated to be free from self-accelerating solutions and to give the above limiting form. Application is made to the scattering of light by the electron and to the rectilinear motion of the electron towards a proton. The theory is entirely classical.

C. Strachan (Aberdeen).

Harish-Chandra. Motion of an electron in the field of a magnetic pole. Physical Rev. (2) **74**, 883–887 (1948).

Recently Dirac [same vol., 817–830 (1948); these Rev. **10**, 345] has revived interest in his theory of the electromagnetic field allowing the existence of free magnetic poles. An electron cannot have bound states in the field of an elementary pole. The present paper extends this work to include the interaction due to the magnetic moment of the electron. There are nevertheless no bound states.

C. Strachan (Aberdeen).

Quantum Mechanics

***Landau, L., and Lifšic, E. Kvantovaya Mehanika. Čast' I. Nerelyativistskaya Teoriya. [Quantum Mechanics. Part I. Nonrelativistic Theory.]** OGIZ, Moscow-Leningrad, 1948. 567 pp.

This book is more nearly a definitive text than an introductory one. Its rather extensive length is devoted strictly to the nonrelativistic theory, so that even spin-orbit coupling is treated only schematically and without presenting the exact form of the operator. The authors have not used the space at their disposal for detailed belaboring of elementary points, mathematical or experimental; their presentation is usually quite adequately clear for readers of suitable background, but is never prolix. They are thus able to cover rather thoroughly all of the main methods and applications of nonrelativistic quantum mechanics, with the single exception of the theory of solids. This is represented only by a half-dozen pages on Bloch's theorem regarding the motion of electrons in periodic fields.

Subjects treated with unusual thoroughness are the quasi-classical case (phase-integral methods), the theory of collisions, and, in particular, the applications of group theory to the discussion of polyatomic molecules. Also, in other parts of the book, there is generally to be found material not usually incorporated in textbooks; for example, in the chapter on spin there is an account of nonrelativistic (three-dimensional) spinors and a proof of Kramers' theorem on degeneracy in electric fields. Only in questions of basic principle, for example, the theory of measurement, is the treatment likely to be less thorough and adequate.

There does not seem to exist in English a book providing the particular values which this one provides for Russian students. *W. H. Furry* (Cambridge, Mass.).

Moyal, J. E. Quantum mechanics as a statistical theory. Proc. Cambridge Philos. Soc. **45**, 99–124 (1949).

Quantum mechanics is reformulated in terms of phase-space distributions. These can have negative values and are only aids to calculation corresponding to the fact that no true distribution functions can exist for the values of non-commuting operators. The author tends to interpret quantum dynamics as an inherently stochastic process. The temporal transformations of the system are represented by integral operators forming a continuous unitary group. Some unresolved discrepancies arise from the fact that, in the theory of discrete Markov processes, unitary transformations correspond to deterministic processes, i.e., preclude the "diffusion" of the probability "fluid." *L. Tisza.*

Viguier, Gabriel. L'équation de Schrödinger pour un oscillateur harmonique linéaire. C. R. Acad. Sci. Paris **226**, 1958–1959 (1948).

L'auteur considère l'équation de Schrödinger de propagation d'un corpuscule de masse m dans un champ où la fonction d'ondes est ψ:

$$\Delta\psi-(8\pi^2m/h^2)V\psi-(4\pi im/h)\partial\psi/\partial t=0,\qquad V=2\pi^2m\omega^2x^2$$

(cas de l'oscillateur linéaire). Comme on sait, il est possible par un changement convenable de ramener cette équation à l'équation linéaire du second ordre:

$$d^2H_k/dq^2-2qdH_k/dq+(\lambda-1)H_k=0,$$

où H_k est un polynome d'Hermite de degré k avec les valeurs propres $\lambda=2k+1$. Les propriétés des polynomes d'Hermite sont réduits enfin par une transformation à certaines propriétés géométriques associées à une équation de Ricatti $d\tau/dq+\tau^2-2q\tau+\lambda-1=0$. *M. Pinl* (Cologne).

Viguier, Gabriel. Notions métriques liées à une vibration moléculaire en quatrième puissance. C. R. Acad. Sci. Paris **227**, 266–268 (1948).

L'auteur a dans des précédentes études [cf. l'analyse ci-dessus] montré qu'il était possible de lier une équation différentielle aussi ancienne que l'équation de Riccati à des notions de quanta et ainsi de faire correspondre divers états énergétiques à des notions métriques classiques, telles que celles des courbes isométriques ou isoradiiques liées à ce type d'équation. Dans la présente note, l'auteur se sert de certains types d'anneaux plans dans lesquels l'énergie potentielle, pour des petits déplacements, est proportionnelle à la quatrième puissance du déplacement, et traite, correspondant à la méthode précédente, le cas d'un oscillateur unidimensionnel pour lequel le potentiel $V(x)=ax^4$.

M. Pinl (Cologne).

Viguier, Gabriel. Enchaînement et quantification: une généralisation du théorème de Darboux à propos du rotateur sphérique. C. R. Acad. Sci. Paris **227**, 504–506 (1948).

L'auteur a dans une précédente étude [cf. les deux analyse ci-dessus] montré, faisant choix du cas particulier de l'oscillateur harmonique linéaire ou de la vibration moléculaire en quatrième puissance, qu'il était possible de faire

correspondre à la théorie des quanta des notions métriques liées à l'équation classique de Riccati et ainsi aborder le discontinu par le continu ou inversement. Dans la présente note, l'auteur traite de la même méthode le rotateur sphérique pour lequel les différents points sont rapporté à la colatitude Θ et la longitude φ par l'application des fonctions sphériques de Laplace $Y_k(\Theta, \varphi)$. *M. Pinl* (Cologne).

Massignon, Daniel. Relations d'indétermination et fluides quantiques. La méthode de l'espace de configuration en théorie cinétique quantique. C. R. Acad. Sci. Paris **228**, 1280–1282 (1949).

Massignon, Daniel. Relations d'indétermination et fluides quantiques. La seconde quantification en théorie cinétique quantique. C. R. Acad. Sci. Paris **228**, 1331–1333 (1949).

Pierucci, Mariano. Una deduzione immediata dell'equazione di Schrödinger da quella di D'Alembert. Atti Sem. Mat. Fis. Univ. Modena **1**, 119–120 (1947).

Averbah, V. L., and Medvedev, B. V. On the theory of quantized space-time. Doklady Akad. Nauk SSSR (N.S.) **64**, 41–44 (1949). (Russian)

The authors investigate the theory of quantized space-time proposed by H. S. Snyder [Physical Rev. (2) **71**, 38–41 (1947); these Rev. **8**, 412]. They regard the method of Snyder as essentially a consideration of the group of motions of a hypersphere in five-space, and from this standpoint explain the arbitrariness in his choice of momentum operators as associated with the well-known arbitrariness in the definition of "distance" in a space of constant curvature. They point out that Snyder's scheme can be obtained uniquely if one imposes certain conditions on the coordinate operators. They also set up a more general scheme, of which Snyder's is a special case, and find that the former, like the latter, is not invariant under translations. The assumption of commuting coordinates is found to lead to results essentially equivalent to those of the usual quantum theory.

The authors treat the field equations considered by Snyder [Physical Rev. (2) **72**, 68–71 (1947); these Rev. **8**, 608] by introducing new variables in the space of which ("the wave-equation space") the equations take on their usual form. However, in this space point charges are found to have extended charge distributions, leading to differences from the usual theory. *N. Rosen* (Chapel Hill, N. C.).

Gel'fand, I. M., and Yaglom, A. M. General Lorentz invariant equations and infinite-dimensional representations of the Lorentz group. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **18**, 703–733 (1948). (Russian)

[A short account of this paper's main results was published in Doklady Akad. Nauk SSSR (N.S.) **59**, 655–658 (1948); these Rev. **9**, 496.] The authors investigate Lorentz invariant equations of the form

$$L^k\partial\psi/\partial x^k+i\varkappa\psi=0, \qquad k=0, \cdots, 3. \tag{1}$$

Here the wave function $\psi(x^0, x^1, x^2, x^3)$ has a finite or an infinite number of components (i.e., it is a vector in a finite- or an infinite-dimensional vector space R), L^k are linear operators on R, and $\varkappa$ is a nonvanishing real constant. To every Lorentz transformation $x^i\to x'^i=l^i{}_kx^k$ corresponds a transformation $\psi\to\psi'=S\psi$ such that the S form a representation D_R of the Lorentz group on R. Then (1) is Lorentz invariant if, for every Lorentz transformation, (2) $L^i=l^i{}_jSL^jS^{-1}$. The authors are mainly concerned with the construction of all systems (L^i) satisfying the relations (2). Their analysis is based on the infinitesimal relations which follow from (2). Let $x^i\to x'^i=x^i+\epsilon^i{}_kx^k=x^i+g^{ij}\epsilon_{jk}x^k$ be an infinitesimal Lorentz transformation ($g^{00}=-g^{11}=-g^{22}=-g^{33}=1$, $g^{ij}=0$ $(i\neq j)$; $\epsilon_{ij}=-\epsilon_{ji}$). Then $\psi\to\psi'+\frac{1}{2}\epsilon_{ik}I^{ik}\psi$ $(I^{ik}=-I^{ki})$, where the I are the infinitesimal generators of the representation D and satisfy the well-known commutation rules for infinitesimal Lorentz transformations. The relations (2) imply

$$[L^i, I^{jk}]=g^{ij}L^k-g^{ik}L^j, \quad i, j, k=0, \cdots, 3, \tag{3}$$

where $[A, B]$ denotes the commutator $AB-BA$. These relations are sufficient to insure (2) if only proper Lorentz transformations are considered while spatial reflections must be treated separately. It is easily shown that the solution of (3) reduces to the solution of the following system:

$$[L^0, I^{jk}]=0 \ (j, k=1, 2, 3); \quad [[L^0, I^{ik}], I^{ik}]=L^0, \tag{4}$$

the remaining operators being defined by $L^k=[L^0, I^{ik}]$ $(k=1, 2, 3)$. The finite-dimensional case has been treated by various writers [cf. the note cited above]. The authors are therefore mainly interested in the infinite-dimensional case. Their methods, however, cover both cases.

For the solution of (4) the form of the infinitesimal operators I^{ik} must be known. It is assumed that the representation D_R may be decomposed into a finite or at most countably infinite number of irreducible representations of the Lorentz group. The irreducible representations (both finite- and infinite-dimensional) may be characterized by a pair of numbers (k_0, k_1), where k_0 is integral or half integral, and k_1 an arbitrary complex number. (The two pairs (k_0, k_1) and $(-k_0, -k_1)$ lead to equivalent representations.) The representation space $R_{(k_0,k_1)}$ may be decomposed into $(2k+1)$-dimensional subspaces invariant under spatial rotations and spanned by vectors ξ^k_p (k integral or half integral; $p=k, k-1, \cdots, -k+1, -k$). In general, k assumes the values $|k_0|, |k_0|+1, |k_0|+2, \cdots$, so that $R_{(k_0,k_1)}$ is infinite-dimensional. If $2k_1$ is integral, has the same parity as $2k_0$, and if $|k_1|>|k_0|$, then $R_{(k_0,k_1)}$ is finite-dimensional, and k assumes the values $|k_0|, |k_0|+1, \cdots, |k_1|-1$. (This corresponds, in usual notation, to the representation $D_{jj'}$, where $j=\frac{1}{2}|k_0+k_1|$, $j'=\frac{1}{2}|k_0-k_1|$.) All unitary representations of the proper Lorentz group are obtained either by letting k_1 be purely imaginary, or by choosing $k_0=0$ and k_1 real such that $0<|k_1|\leq 1$. For a given pair (k_0, k_1), the infinitesimal generators of the corresponding irreducible representation are determined by

$$I^{23}\xi^k_p=ip\xi^k_p, \quad (I^{31}+iI^{12})\xi^k_p=\{(k+p)(k-p+1)\}^{\frac{1}{2}}\xi^k_{p-1},$$
$$(I^{31}-iI^{12})\xi^k_p=-\{(k+p+1)(k-p)\}^{\frac{1}{2}}\xi^k_{p+1};$$
$$2I^{01}\xi^k_p=-\{(k+p)(k-p)\}^{\frac{1}{2}}B_k\xi^{k-1}_p+pA_k\xi^k_p$$
$$+\{(k+p+1)(k-p+1)\}^{\frac{1}{2}}B_{k+1}\xi^{k+1}_p,$$

with $A_k=2k_0k_1/k(k+1)$, $B_k=\{(k^2-k_0^2)(k^2-k_1^2)/k^2(k^2-\frac{1}{4})\}^{\frac{1}{2}}$. (The sign of B_k may be chosen arbitrarily.)

The form of the operator L^0 [cf. (4)] is determined in a straightforward way. Denote by $R_\tau=R_{(k_0,k_1)}$ the irreducible subspaces of R invariant under the transformations S, and by $\xi^k_{p\tau}$ the vectors which span R_τ. The authors find that (1) $L^0\xi^k_{p\tau}=\sum_{\tau'}c^k_{\tau\tau'}\xi^k_{p\tau'}$, (2) $c^k_{\tau\tau'}$ can be different from zero only if the pair (k_0', k_1') is equivalent to (k_0+1, k_1), (k_0-1, k_1), (k_0, k_1+1), or (k_0, k_1-1) (if $\tau\sim(k_0, k_1)$, and $\tau'\sim(k_0', ik_1')$), (3) the coefficients $c^k_{\tau\tau'}$ are uniquely determined by constants $c_{\tau\tau'}$ which may be arbitrarily chosen (for example, the authors obtain $c^k_{\tau\tau'}=c_{\tau\tau'}\{(k+k_0+1)(k-k_0)\}^{\frac{1}{2}}$ if $k_0'=k_0+1$, $k_1'=k_1$, and similar expressions in the remaining cases). If

spatial reflections are included, the representation D_R contains, with every irreducible representation (k_0, k_1), also, the representation $(k_0, -k_1)$ (unless $k_0=0$ or $k_1=0$), and additional conditions on the coefficients $c_{\tau\tau'}$ are established.

The authors next determine all systems (1) which may be derived from a variational principle with an invariant real Lagrangian of the form

(5) $L=(2i)^{-1}\{(L^k\partial\psi/\partial x^k, \psi)-(\psi, L^k\partial\psi/\partial x^k)\}+\kappa(\psi, \psi)$.

Here (ψ_1, ψ_2) denotes a Hermitian form invariant under the transformations S, so that (6) $(S\psi_1, S\psi_2)=(\psi_1, \psi_2)$ for every S and any two vectors ψ_1, ψ_2 in R, and furthermore

(7) $(L^k\psi_1, \psi_2)=(\psi_1, L^k\psi_2), \qquad k=0, \cdots, 3.$

The analysis of (6) is similar to that of (2) (involving again the infinitesimal operators I^{kl}) and leads to explicit expressions for the coefficients of the Hermitian form referred to the basic vectors $\xi^k_{\sigma\nu}$, while the combination of (6) and (7) yields further conditions on the $c_{\tau\tau'}$. From the Lagrangian (5) one obtains a current vector (8) $s^k=(\epsilon/\hbar)(L^k\psi, \psi)$ and an energy-momentum tensor

(9) $T_l^k=(2i)^{-1}\{(L^k\partial\psi/\partial x^l, \psi)-(\psi, L^k\partial\psi/\partial x^l)\}$,

both satisfying conservation laws.

The possible values of rest mass and spin are obtained by inserting in (1) a plane wave of the form $\psi=\psi_{(0)}\exp(-ip_0x^0)$ (describing a particle in its rest system), where $\psi_{(0)}$ does not depend on the coordinates x^k. The authors conclude that the rest masses are $m_i=(\kappa c/\hbar)(1/\lambda_i)$ (λ_i the eigenvalues of L^0), and that the possible spin values are those k for which the submatrix $\|c^k_{\tau\tau'}\|$ (in τ, τ') has nonvanishing eigenvalues. [In the reviewer's opinion, this discussion is inadequate because the equation (1), in general, admits solutions $\psi_{(0)}\exp(-ip_kx^k)$, where p_k is a space-like or a null vector, for which no rest system can be defined. Cf. also the following review.]

In the last section several examples are discussed, both for finite- and infinite-dimensional R. The authors treat only the simplest infinite-dimensional cases, viz., those where D_R itself is irreducible. There are two possibilities: (I) $\tau\sim(0, \frac{1}{2})$, and (II) $\tau\sim(\frac{1}{2}, 0)$. In both cases $L^0\xi^k_\sigma=(k+\frac{1}{2})\xi^k_\sigma$, where k runs through all integral or all half integral values respectively, and in both cases (1) is derivable from a Lagrangian (cf. (5)), the invariant Hermitian form being the unit form. [Reviewer's note: These two cases have already been discussed by E. Majorana, Nuovo Cimento (N.S.) 9, 335–344 (1932).] *V. Bargmann.*

Gel'fand, I. M., and Yaglom, A. M. On Lorentz invariant equations to which correspond a definite charge and a definite energy. Doklady Akad. Nauk SSSR (N.S.) 63, 371–374 (1948). (Russian)

Gel'fand, I. M., and Yaglom, A. M. Pauli's theorem for general Lorentz invariant equations. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 18, 1096–1104 (1948). (Russian)

The first of these two papers merely gives the main results; the second contains the detailed proofs. W. Pauli [Physical Rev. (2) 58, 716–722 (1940)] has proved the following theorem concerning Lorentz invariant equations for wave functions with a finite number of components. For integral spin values neither the charge density nor the total charge of the system described may be (positive or negative) definite, for half integral spin values neither the energy density nor the total energy may be definite, provided the wave equations are differential equations and the expressions for charge and energy density are obtained by differential operations. The authors generalize this theorem to wave functions with an infinite number of components which satisfy the equations studied in a previous paper, more specifically, those equations which are derived from an invariant Lagrangian [cf. the preceding review, to which the reader is referred for details]. Charge and energy density are given by s^0 and T_0^0, respectively [cf. equations (8) and (9) of the preceding review]. By discussing the irreducible representations $\tau\sim(k_0, k_1)$ (where $k_1=k_1'+ik_1''$) which occur in D_R, the authors establish the following results (in any one of the following statements charge (energy) may be interpreted as either charge (energy) density or total charge (energy)). (A) If, for some τ, $k_1''\neq 0$, and $2k_1'$ is not integral, neither charge nor energy is definite. (B) If, for some τ, $k_1''\neq 0$, the charge is indefinite for an integral k_1', and the energy is indefinite for a half integral k_1'. (C) If, for some τ, k_1 is real, but $2k_1$ is not integral, the charge is indefinite for an integral k_0, and the energy is indefinite for a half integral k_0. (D) If, for some τ, both k_0 and k_1 are integral, the charge is indefinite; if both k_0 and k_1 are half integral, the energy is indefinite. There remains the possibility that of the two numbers k_0, k_1 one is integral and the other half integral, as in the two cases (I) and (II) [cf. the preceding review]. The authors assert that in these two cases both charge and energy density are definite. [Reviewer's note. While the assertion concerning the charge density is correct, one can construct solutions with both positive and negative energy densities or total energies. The authors base their proof on the decomposition of the general solution of the wave equation into plane waves with time-like wave vectors; the wave equation, however, also admits plane wave solutions whose wave vectors are space-like or null vectors.]

V. Bargmann (Princeton, N. J.).

Gel'fand, I. M., and Yaglom, A. M. Charge conjugation for general Lorentz invariant equations. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 18, 1105–1111 (1948). (Russian)

This paper deals with the applicability of the method of charge conjugation [developed for Dirac's equation by E. Majorana, Nuovo Cimento (N.S.) 14, 171–184 (1937); and H. A. Kramers, Nederl. Akad. Wetensch., Proc. 40, 814–823 (1937)] to the general Lorentz invariant equations studied by the authors in a previous paper [see the second preceding review]. In the absence of an electromagnetic field the equations read $L^k\partial\psi/\partial x^k+i\kappa\psi=0$, where ψ is a vector with a finite or an infinite number of components and κ is a real constant. For an external electromagnetic field given by the four-vector potential φ_k the authors set

(*) $L^k(\partial\psi/\partial x^k-i\epsilon\varphi_k\psi)+i\kappa\psi=0.$

The charge conjugate wave function ψ^Q is defined by an antilinear transformation $\psi^Q=Q\psi$ (i.e., $Q(\psi_1+\psi_2)=Q\psi_1+Q\psi_2$, and $Q(\lambda\psi)=\lambda^*Q\psi$), and it is assumed that (a) ψ^Q satisfies the equation (*) with ϵ replaced by $(-\epsilon)$, and (b) $(\psi^Q)'=(\psi')^Q$, where $\psi'=S\psi$ is the Lorentz transformed wave function. (The condition (b) expresses the Lorentz invariance of charge conjugation.) The condition (a) implies $QL^k+L^kQ=0$, while (b) implies $SQ=QS$. Using the results reviewed above, the authors determine the possible forms of Q. It is found (1) that the operator Q cannot always be defined, since the constants $c_{\tau\tau'}$ of the previous paper must satisfy certain additional conditions, (2) that Q (if it can be defined at all) is (up to a factor) uniquely determined if the equation (*) is irreducible. *V. Bargmann* (Princeton, N. J.).